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Devoted to Science and the Mechanic Arts

Vol. 200 NOVEMBER, 1925 No. 5

MOTION OF ELECTRONS IN GASES.*

BY

J. S. TOWNSEND, M.A., F.R.S.

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(1) IN THE earlier work on the properties of ions in gases there are many investigations which indicate that the masses associated with the atomic charges are large compared with the mass of a molecule of a gas.

This result was obtained from measurements of the velocity of the ions in the direction of the electric force, and of the rate of diffusion of the ions. In air at atmospheric pressure the velocity of the positive ions was found to be about 1.4 centimetres a second under a force of one volt per centimetre, and that of the negative ions 1.7 centimetres per second. These velocities are smaller than the velocity which a molecule of the gas would acquire if it has a charge equal to that of a monovalent ion in an electrolyte. Also the rate of diffusion of the ions was found to be smaller than the rate of diffusion of molecules. In air at atmospheric pressure, the coefficient of diffusion of the positive ions was found to be .03 and that of the negative ions .043, which shows that the masses of the ions in air are larger than the mass of a molecule of carbonic acid, as the coefficient of diffusion of the latter into air is .142.

From later experiments, it was found that in gases at pressures of a few centimetres of mercury the velocity W in the direc-

* Address delivered Friday, September 19, 1924, on the occasion of the celebration of the centenary of the founding of The Franklin Institute.

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tion of the electric force and the coefficient of diffusion D becomes very large, and the rate of increase of W and D as the pressure was reduced was much greater than the increase of the inverse of the pressure. These changes indicate a reduction in the mass associated with the atomic charge, and occur when electrons move freely in the gas and do not become attached to molecules. The motion which is characteristic of free electrons may be obtained in any gas by reducing the pressure and increasing the electric force, but in the absence of water vapor, and other impurities which tend to form ions, free electrons are obtained in some gases at several centimetres pressure with small electric forces of the order of one or two volts per centimetre.

(2) The most direct method of investigating the motion of electrons through a gas, is to determine experimentally the velocity of agitation U and the velocity W in the direction of the electric force. From the results of these experiments the effect of a collision with a molecule on the motion of an electron may be estimated by methods similar to those used in the kinetic theory of gases.

The principle of the method which has been used to find the velocity of agitation of the electrons consists of measuring the divergence of a stream of electrons moving in a uniform electric field. The first experiments were made with an apparatus in which a stream of electrons passed through a circular aperture in a metal sheet, and moved under an electric force to the receiving electrodes consisting of a disc opposite the aperture and a large guard ring in the same plane as the disc. With a constant electric force Z the ratio of the charge received by the disc to the charge received by the ring diminished as the gas pressure p was reduced, which indicates that the velocity of agitation of the electrons increases when the ratio Z/p is increased.[1]

When electrons move under the action of an electric force their velocity of agitation approaches a constant final value which is attained when the energy lost in collisions with molecules is equal to the energy gained by moving in the direction of the electric force. In the steady state of motion the energy of agitation of the electrons may be much greater than that of the molecules of the gas. This property of the electrons was first observed in air which was ionized by Röntgen rays.

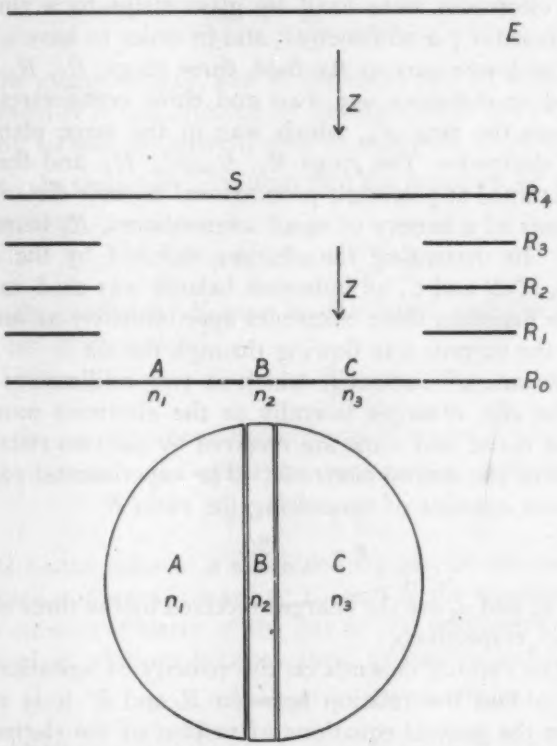
Similar results were also obtained in air ionized by a radio-

[1] *Proc. Roy. Soc.*, A, **81**, p. 464, 1908.

active substance, and with electrons set free from a metal surface by the action of ultra-violet light.[2]

(3) With the apparatus which was first used, where the central receiving electrode was a disc, it was not possible to measure the velocity of the electrons in the direction of the electric force, so that another apparatus of the form shown in Fig. 1 was

FIG. 1.



adopted, which was suitable for the measurements of both the velocities U and W.

In this apparatus a beam of ultra-violet light falls obliquely on the metal plate E, and the electrons which are set free move under an electric force Z towards a metal sheet fixed in metal ring R_4. A rectangular slit, S, 1.5 centimetres long and two millimetres wide, was cut in the metal sheet, so that a narrow stream of

[2] C. E. HASSELFOOT, *Proc. Roy. Soc.*, **82**, A, p. 18, 1909, and **87**, A, p. 350, 1912.

electrons passed through the slit and moved under a similar force Z to the receiving electrodes A, B and C, which are in a plane four centimetres from the slit.

The outer electrodes A and C were segments of a disc seven centimetres in diameter, and the central electrode B a strip 4.5 millimetres wide and seven centimetres long, insulated from the segments on each side of it by gaps half a millimetre wide. The receiving electrodes were fixed by glass strips to a ring R_0 of internal diameter 7.2 centimetres; and in order to have a uniform force in the lower part of the field, three rings, R_1, R_2, and R_3, were fixed at distances one, two and three centimetres, respectively, from the ring R_0, which was in the same plane as the receiving electrodes. The rings R_1, R_2, R_3, R_4, and the plate E were maintained at potentials proportional to their distances from R_0 by means of a battery of small accumulators, R_0 being at zero potential. In measuring the charges received by the insulated electrodes, A, B and C, an induction balance was used, and it was possible to maintain these electrodes approximately at zero potential while the current was flowing through the slit.

The stream of electrons, which is two millimetres wide in passing the slit, diverges laterally as the electrons move under the electric force, and some are received by the two electrodes on either side of the central electrode. The experimental part of the investigation consists of measuring the ratio R,

$$R = \frac{n_2}{n_1 + n_2 + n_3}$$

where n_1, n_2 and n_3 are the charges received by the three electrodes A, B and C, respectively.

(4) The ratio R depends on the velocity of agitation U, and in order to find the relation between R and U it is necessary to consider the general equations of motion of the electrons. As the electrons traverse four centimetres from the plate E to the slit S, it may be assumed that they attain the steady state of motion corresponding to the force Z before passing through the slit. In moving from the slit to the receiving electrodes, the motion remains uniform and the velocity of agitation U is the same at all points.

Let n be the number of electrons per cubic centimetre at any point x y z in the space between the slit and the receiving electrodes, and P their partial pressure. Taking the axes of x and y

in the plane of the slit and z in the direction of the electric force with the origin at the centre of the slit, and nu, nv and nw the numbers of electrons crossing unit areas normal to the axes per second, the following equations of motion are obtained.

$$\begin{aligned} \frac{Pu}{K} &= -\frac{dP}{dx} \\ \frac{Pv}{K} &= -\frac{dP}{dy} \\ \frac{Pw}{K} &= -\frac{dP}{dz} + nZe \end{aligned} \tag{1}$$

e being the atomic charge, and K the coefficient of diffusion.

The ratio R is not affected by fluctuations in the intensity of the stream, so that the current may be supposed to be constant at any point. In this case $\frac{dn}{dt}$ is zero and the equation of continuity may be written

$$\frac{d(Pu)}{dx} + \frac{d(Pv)}{dy} + \frac{d(Pw)}{dz} = 0 \tag{2}$$

Thus the coefficient K may be eliminated and the following equation for P is obtained

$$\nabla^2 P = Ze\frac{dn}{dz} \tag{3}$$

Since the partial pressure P is $\frac{1}{3}mnU^2$ where U^2 is the mean square of the velocity of agitation, equation (3) reduces to

$$\nabla^2 n = \frac{3Ze}{mU^2} \times \frac{dn}{dz} \tag{4}$$

Let M be the mass of a molecule of a gas, Ω^2 the mean square of its velocity of agitation at 15° C., and N the number of molecules per cubic centimetre of the gas at 760 millimetres pressure. The following relations between these quantities and the atomic charge e may be used to simplify the coefficient of $\frac{dn}{dz}$ in equation (4).

$$\frac{1}{3}MN\Omega^2 = 1.01 \times 10^6 \text{ dynes}, \quad Ne = 1.22 \times 10^{10} \text{ E.S. units.}$$

The energy of agitation of the electrons may be expressed in terms of the energy of agitation $\frac{1}{2}M\Omega^2$ of a molecule of a gas at 15° C. by the relation

$$mU^2 = kM\Omega^2$$

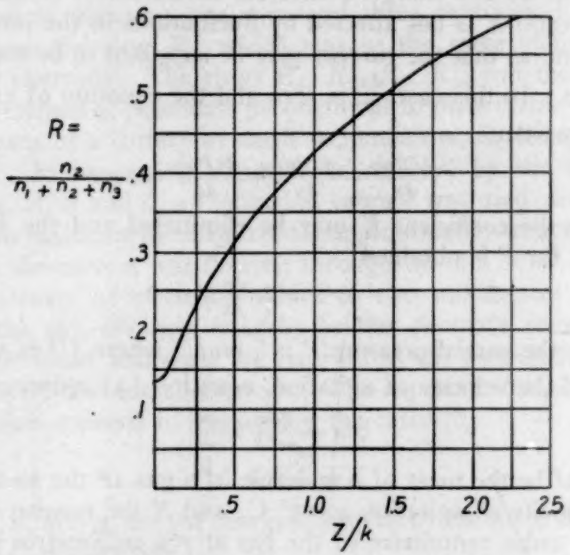
Thus $\frac{3Ze}{mU^2}$ becomes $\frac{3Ze}{kM^2}$, which reduces to $\frac{Z}{k} \times 40.3$ when Z is expressed in volts per centimetre.

Thus equation (4) reduces to

$$\nabla^2 n = 40.3 \times \frac{Z}{k} \times \frac{dn}{dz} \qquad (5)$$

This equation shows that n depends only on the ratio Z/k, and a solution of equation (5) may be found which satisfies the boundary conditions of the apparatus and gives n as a function of Z/k. The value of $R = n_2/(n_1 + n_2 + n_3)$ may be obtained by integration over the surface of the electrodes. The expression thus found for R is complicated, and it is convenient to represent

FIG. 2.



it by means of a curve, where the ordinates are R and the abscissæ the values of the ratio Z/k.

In order to calculate the value of R corresponding to a value of Z/k, a comparatively simple formula [3] may be used to find the larger values of R, but for the smaller values of R, it is necessary to express the solution of equation (5) in the form of a series. The values of R thus obtained are given by the curve, Fig. 2, in terms of Z/k.

The calculations of R from the solution of equation (5), where n is expressed in the form of a series, were made by Mackie,[4] and

[3] J. S. TOWNSEND and H. T. TIZARD, *Proc. Roy. Soc.*, A, **88**, p. 336.
[4] J. H. MACKIE, *Proc. Roy. Soc.*, **90**, p. 69, 1914.

his figures were subsequently confirmed when it was found necessary to make a correction for an apparatus [5] where the centre of the stream did not fall exactly on the centre of the electrode E.

The curve (Fig. 2) gives the values of R when the apparatus is arranged so that the centre of the stream falls on the centre of the electrode B (Fig. 1), the charge n_1 being in this case equal to n_3.

(5) The theory up to this point applies either to streams of ions or streams of electrons, as no particular value has been attributed to the mass m. But the following considerations show that considerable differences may be obtained in the values of k depending on the mass associated with the atomic charge. When the charged particles attain the final steady motion the energy they acquire in moving in the direction of the force is equal to the energy they lose in collision with molecules. The loss of energy in a collision depends on the mass of the particle and the amount by which its energy of agitation exceeds that of the molecules of the gas. Thus when an ion makes a large number of collisions with molecules, the steady motion is attained when the energy of agitation of the ions exceeds that of the molecules by a comparatively small amount, and the factor k is approximately equal to unity. In this case Z/k does not differ appreciably from Z, and the theory shows that the ratio R is a function Z which is the same for all gases and is independent of the pressure. This conclusion agrees with the results obtained experimentally with moist gases. The values of R for different forces were found to be the same as those given by the curve (Fig. 2) when $k = 1$, and for a given force Z the same value of R was obtained for different gases over a large range of pressures.

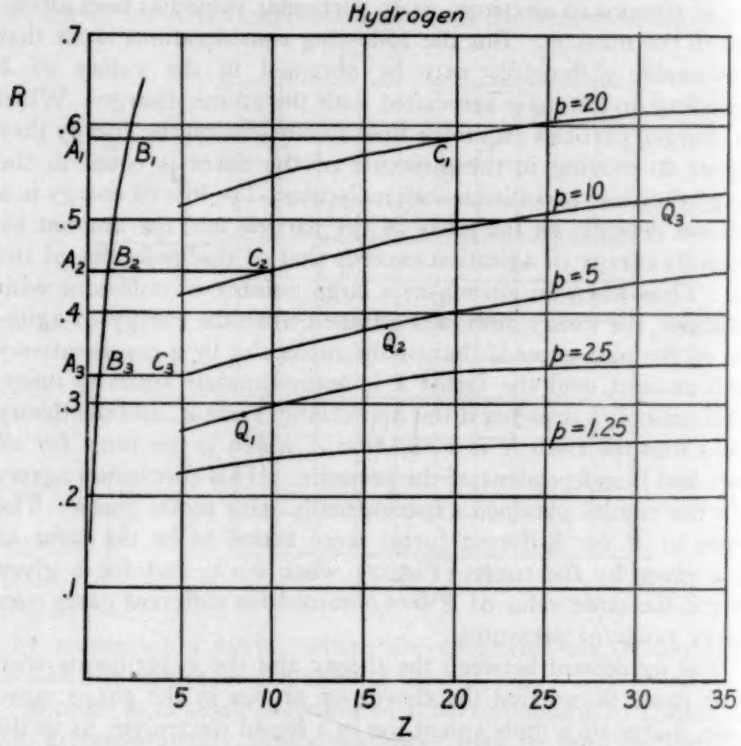
The agreement between the theory and the experiments with moist gases shows that the charge on an ion in the gas is equal to the charge on a monovalent ion in a liquid electrolyte, as in the calculations the value of Ne was taken to be 1.22×10^{10}.

(6) The values of R obtained in dry gases are much less than in moist gases for certain ranges of forces and pressures. Also in dry gases, the divergence obtained with a given electric force increases as the pressure is reduced. Thus the value of k is large and increases as p is reduced. In these cases the loss of energy by collision with molecules does not become equal to the energy

[5] J. S. TOWNSEND and V. A. BAILEY, *Phil. Mag.*, 44, p. 1033, 1922.

acquired by moving in the direction of the electric force until the kinetic energy of the charged particles is much greater than that of molecules of the gas. This result shows that the mass of the charged particle must be small compared with that of a molecule, and it may be assumed that the electrons move freely in the gas. It will be seen that this conclusion is in accordance with the determinations of the velocity in the direction of the electric force.

FIG. 3.



Also, it follows from the theory of the motion in the steady state, that k is a function of the ratio Z/p. There are large differences in the values of k depending on the gas, as the loss of energy in collisions depends on the mass and other physical properties of molecules.

The curves, Fig. 3, give the results of the determinations of R in hydrogen. Each curve corresponds to a definite pressure, the ordinates being the values of $R = n_2/(n_1 + n_2 + n_3)$ and the

abscissæ the electric force Z in volts per centimetre. The curve $B_1\ B_2\ B_3$ is the theoretical curve (Fig. 2), the abscissæ in this case being Z/k. The curves representing the experimental determinations show that for a given force Z, R diminishes as the pressure of the hydrogen is reduced from 20 to 1.25 millimetres.

When R is determined experimentally, the value of Z/k is found from the theoretical curve and the value of k is thus obtained. For example, with the gas at 20 millimetres pressure, the ratio R is found to be .58 with the electric force of 20 volts per centimetre as shown by the point C_1. If A_1C_1 be drawn parallel to the axis of Z, cutting the theoretical curve at B_1, the length A_1B_1 represents Z/k and the value of k is A_1C_1/A_1B_1.

At the points C_2 and C_3 on the curves for the pressures $p = 10$ and $p = 5$, corresponding to the electric forces $Z = 10$ and $Z = 5$, the ratio of Z/p is the same as at C_1. The values of k obtained in a similar manner are A_2C_2/A_2B_2 and A_3C_3/A_3B_3 and it is seen that

$$\frac{A_1C_1}{A_1B_1} = \frac{A_2C_2}{A_2B_2} = \frac{A_3C_3}{A_3B_3} = 9.3$$

Hence in hydrogen the value of k is 9.3 when $Z/p = 1$.

The experiments, therefore, are consistent in showing that k depends only on the ratio Z/p.

(7) A comparison of the divergencies of streams of electrons in different gases is shown by the curves, Fig. 4, the ordinates being the values of R and the abscissæ the forces Z. As in Fig. 3, the theoretical curve B is also given, the abscissæ in this case being Z/k. In helium, nitrogen and hydrogen at 20 millimetres pressure, the streams of electrons are more divergent in nitrogen than in hydrogen and more divergent in helium than in nitrogen.

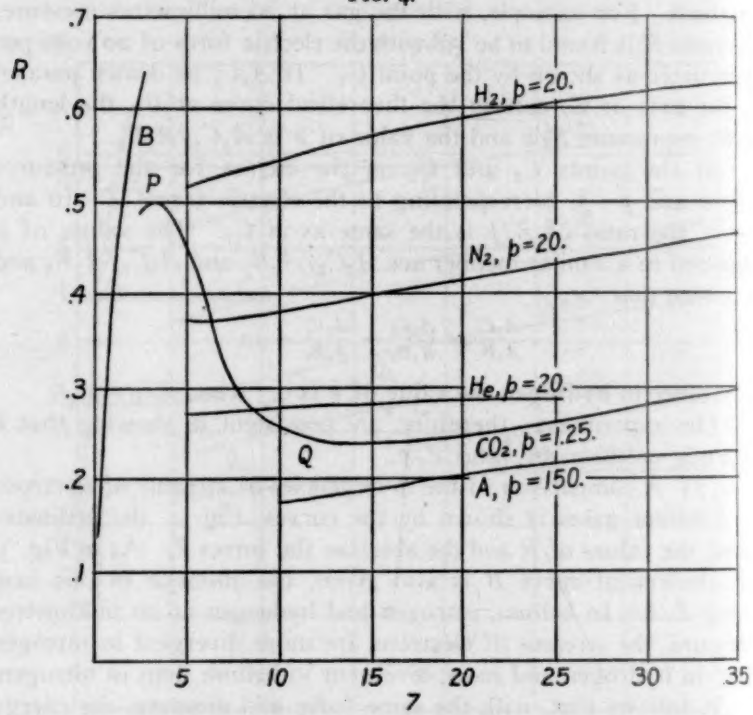
It follows that, with the same force and pressure, the energy of agitation of electrons is greater in helium than in nitrogen and greater in nitrogen than in hydrogen.

The curve representing the values of R in carbonic acid at 1.25 millimetres pressure shows that as the force increases from about three volts per centimetre at the point P to about 12 volts per centimetre at Q, there is a great increase in the divergence of the stream, indicating a large rate of increase of k with the electric force. If this gas were at 20 millimetres pressure, the corresponding curve would be much above the curve for hydrogen at 20 millimetres pressure.

The curve for argon at 150 millimetres pressure shows that in argon the values of k are much greater than in other gases, though the values at Z/p are smaller.

It is convenient in order to obtain accurate measurements of k and W, not to have the streams so divergent that the values of R are less than about .25. For this reason the experiments with the

FIG. 4.



monatomic gases were made with much higher gas pressures than with ordinary gases. Also it was found desirable to modify the apparatus by bringing the receiving electrodes (A, B and C, Fig. 1) nearer to the slit S, in order to obtain larger values of R, and most of the experiments with argon and neon were made with an apparatus where the receiving electrodes were two centimetres from the slit.[8]

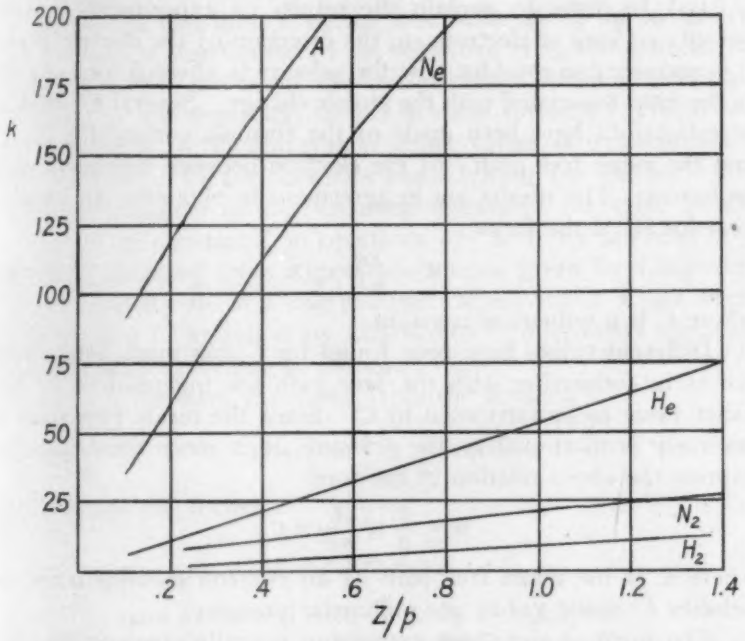
(8) The results of the experiments with each gas may be represented by a curve giving k in terms of the ratio Z/p. The

[8] J. S. TOWNSEND and V. A. BAILEY, *Phil. Mag.*, **44**, p. 1033, 1922.

parts of the curve for hydrogen, nitrogen, helium, neon, and argon for values of Z/p from .1 to 1.4 are given in Fig. 5.

The velocity of agitation U of the electrons may be found from the value of k by the relation $mU^2 = kM\Omega^2$ since the ratio of the mass M of a molecule to the mass m of an electron, and the

FIG. 5.



mean square of the velocity of agitation of a molecule Ω^2 at 15° C. are known.

The value of U is thus found to be

$$U = 1.15 + 10^7 \times \sqrt{k} \text{ (cm. per sec.)} \qquad (6)$$

For some purpose it is convenient to represent the velocity U in terms of the potential difference V required to impart the energy $\frac{1}{2}mU^2$ to the electron.

The following equations give V in terms of U and k.

$$\frac{mU^2}{2} = \frac{kM\Omega^2}{2} = \frac{eV}{300}$$

where V is expressed in volts.

Since $\frac{MN\Omega^2}{3} = 1.01 \times 10^6$, and $Ne = 1.22 \times 10^{10}$, the relation between V and k is found to be $V = \frac{k}{27}$.

Hence when k is 27, the velocity of agitation is 6×10^7 cm. per second, and this is the velocity acquired by the electron in moving between two points differing in potential by one volt.

(9) In order to explain the results of experiments on the velocity of ions or electrons, in the direction of the electric force, it is necessary to consider how the velocity is affected by a change in the mass associated with the atomic change. Several theoretical investigations have been made of the formula connecting W, U and the mean free path l of the electron between collisions with molecules. The results are in agreement in obtaining an expression for W of the form

$$W = \frac{Zel}{mU} \times C \tag{7}$$

where C is a numerical constant.

Different values have been found for C, but many interesting points in connection with the free path are independent of the exact value to be attributed to C. Since the mean free path is inversely proportional to the pressure, it is more convenient to express the above relation in the form

$$W = \frac{Z}{p} \times \frac{eL}{mU} \times C \tag{8}$$

where L is the mean free path of an electron moving with the velocity U in the gas at one millimetre pressure.

The form of the above expression is easily obtained by supposing all the free paths to be equal to the mean free path l, and the velocity agitation to be constant and equal to the mean velocity U. The velocity W may also be taken as being small compared with U. In the intervals between collisions, the acceleration of the electron in the direction of the electric force is Ze/m and the average distance S which it moves in this direction in the time l/U between two collisions is

$$S = \frac{1}{2} Ze \times \frac{l^2}{U^2} \tag{9}$$

Assuming that after a collision with a molecule all directions of motion of the electron are equally probable the velocity W is

$$W = \frac{SU}{l} = \frac{1}{2} \frac{Ze}{m} \times \frac{l}{U} \tag{10}$$

Since the velocity in the direction of Z is zero at the beginning and $2W$ at the end of a free path, the average increase of energy of an electron in traversing a free path is $2mW^2$, and in order that the energy of agitation may remain constant this must be the average energy lost in a collision. Thus the proportion λ of its energy which an electron loses in a collision is $4W^2/U^2$.

When the distribution of the free paths about the mean free path l, and the distribution of the velocities of agitation about the mean velocity U are taken into consideration, different values are obtained for the numerical factors in these formulæ. The principal difficulties in obtaining their exact values is in ascertaining the distribution of the velocities and allowing for the fact that l is not independent of U. In these researches the value usually taken for the constant C in equations (7) and (8) has been .815, which is obtained from a general formula given by Langevin[7] for the velocity W of a charged particle moving in a gas where the velocities of agitation are assumed to be distributed about the mean velocity according to Maxwell's law.

Thus L has been calculated from the formula

$$W = \frac{Z}{p} \times \frac{e}{m} \times \frac{L}{U} \times .815 \qquad (11)$$

and λ from the formula

$$\lambda = 2.46 \times \frac{W^2}{U^2} \qquad (12)$$

If the mass associated with the atomic charge were a molecule of nitrogen, the mean free path and the velocity of agitation the same as for a molecule, the velocity W of the charged particle under a force of one volt per centimetre in nitrogen at atmospheric pressure would be approximately 5 centimetres per second.

If the charged particle were an electron, its energy of agitation equal to that of a molecule of a gas at 15° C. ($k = 1$), and its free path estimated on the hypothesis that its dimensions are negligible compared with that of a molecule, its velocity W under a force of one volt per centimetre in nitrogen at atmospheric pressure would be approximately 6000 centimetres per second.

A theoretical investigation of the motion of electrons in a gas has also been made by Pidduck, and he finds that the distribution of the velocities of the electrons is represented more accurately by

[7] P. LANGEVIN, *Ann. de Chim. et de Phys.*, **28**, p. 336, 1903.

a law given by Lorentz than by Maxwell's law. The value he obtains for the constant C in equation (8) is .92. The theory was also applied to the case where the collisions of electrons with molecules were treated as if the molecules were imperfectly elastic spheres. If δ be the coefficient of restitution and $f = \frac{1}{2}(1 + \delta)$, the following equation to determine f has been obtained by Pidduck:

$$k - f + \frac{kM}{m}(1 - f) = \frac{W^2}{\Omega^2} \qquad (13)$$

where M is the mass of a molecule of the gas through which the electrons move, $M\Omega^2/2$, the mean energy of agitation of a molecule, $kM\Omega^2/2$ the mean energy of agitation of the electrons, and W the velocity of the electrons in the direction of the electric force. Thus for perfectly elastic spheres, $k - 1 = \frac{W^2}{\Omega^2}$.

In the definition of k as it occurs in equation (5), Ω is the velocity of agitation at 15° C., and as this is approximately the temperature at which the experiments were conducted, the values of k which were obtained experimentally may be used in Pidduck's formula.[8]

It is to be noticed that when an electron starts to move under an electric force Z with a small initial velocity of agitation, the velocity in the direction of the electric force is at first comparatively large, but diminishes as the energy of agitation increases. The final value W is attained when U attains its final constant value depending on the ratio Z/p. It is therefore necessary to allow the electrons to move for some considerable distance under the electric force before the velocity W can be measured.

(10) Some evidence of the possibility of obtaining free electrons with comparatively small forces was found by Langevin[9] in an investigation of the velocities of ions in air at various pressures. Under a force of volt per centimetre the velocity in the direction of the electric force of the negative ions was found to be 1.7 centimetres a second in air at 760 millimetres pressure, and 21.9 centimetres to a second at 75 millimetres pressure. If the mass of the ion were unaffected by the reduction in pressure the latter velocity would be 17 instead of 21.9.

[8] F. B. PIDDUCK, *Proc. Roy. Soc.*, A, **88**, p. 296, 1913, and *Proc. Lon. Math. Soc.*, Series 2, **15**, pt. 2, p. 89, 1915.

[9] P. LANGEVIN, *Comptes Rendus*, **134**, p. 646, 1902; *Ann. de Chim. et de Phys.*, (7) **28**, 1903.

As the experiments on the divergence of streams of negative electricity in dry air showed that it is possible to have free electrons moving through several centimetres of the gas, the velocity in the direction of the electric force should, under similar conditions, become very large.

This result was obtained by Lattey [10] in a series of experiments with dry air, where it was found that the velocity of the positive ions was proportional to Z/p, W being $1120 \times Z/p$, for a certain range of forces and pressures; but with negative ions the velocity was much larger than when expressed in the form $W = b\frac{Z}{p}$, the factor b increased with Z/p. For example, in air at 14.3 millimetres pressure the velocity under a force of .57 volt per centimetre was found to be 107 centimetres per second, which gives 2680 as the value of b; and at the same pressure when the force was 1.14 volts per centimetre the velocity was 1126 centimetres per second, which gives 14,000 as the value for b.

Similar results were obtained by Lattey and Tizard [11] with hydrogen and carbonic acid, the large velocities of the negative ions being obtained with pressures up to 200 millimetres in hydrogen with electric forces of about two volts per centimetre.

The method of determining the velocities which was used by Lattey and subsequently by Lattey and Tizard depended on the measurement of the distance a group of ions or electrons move in the space between two gauzes under the action of a constant force which is applied in opposite directions for definite intervals of time. It is a modification of the method used by Rutherford to determine the velocity of ions by an alternating force between parallel plate-electrodes.

The experiments made by Lattey and Tizard are of importance, as they show that when electrons tend to form ions with molecules of the gas, or with some impurity in the gas, the rate of formation of the ions may be reduced either by lowering the pressure or by increasing the electric force. The method of measuring the velocities was, however, not suitable for measurements of the large velocities of electrons in cases where the ratio Z/p is much greater than .1, as in the experiments on the determination of k. It will be seen from other investigations that a stage is reached with the larger values of Z/p when there are no ions in the stream

[10] R. T. LATTEY, *Proc. Roy. Soc.*, A, **84**, p. 173, 1910.

[11] R. T. LATTEY and H. T. TIZARD, *Proc. Roy. Soc.*, A, **86**, p. 349, 1912.

of electrons and the factor b in the expression (bZ/p) for the velocity diminishes as Z/p increases. This is due to the fact that the velocity of agitation increases with the ratio Z/p and the value of b. as indicated by equation (11), is inversely proportional to U.

The velocities of ions in argon and nitrogen were investigated by Franck,[12] who used Rutherford's method, and the results are given in expressions of the form b_1Z and b_2Z for the velocities of positive and negative ions for the gases at atmospheric pressure. In the pure gases the velocities of the negative ions were comparatively large, the velocity b_2 corresponding to one volt per centimetre being 206 for argon and 144 for nitrogen. When the gases contained a small proportion of oxygen, about 1.2 per cent., the velocities became comparatively small, b_2 being reduced to 1.7 in argon and 1.84 in nitrogen. It is difficult to draw any general conclusion from these results, as in a previous investigation by Franck and Pohl the velocity obtained for negative ions in helium was comparatively small, the value of b_2 being 6.31. There is a considerable difference between these results and those obtained by Lattey from experiments with air, where free electrons were obtained in a gas containing a large proportion of oxygen.

The latter result has also been obtained in other investigations of the motion electrons in gases, and velocities in argon and nitrogen have been obtained which are about ten or twenty times as great as those indicated by the values of b_2 obtained by Franck.

(11) In order to make accurate measurements of the velocity W under the exact same conditions as the measurements of k, the apparatus with the electrodes arranged as shown in Fig. 1, was used. Two large coils were set up on either side of the vessel containing the electrodes so that when a current flowed through the coils the stream of electrons was deflected by the magnetic force as shown in the diagram (Fig. 6), the direction of the magnetic force being normal to the plane of the figure.

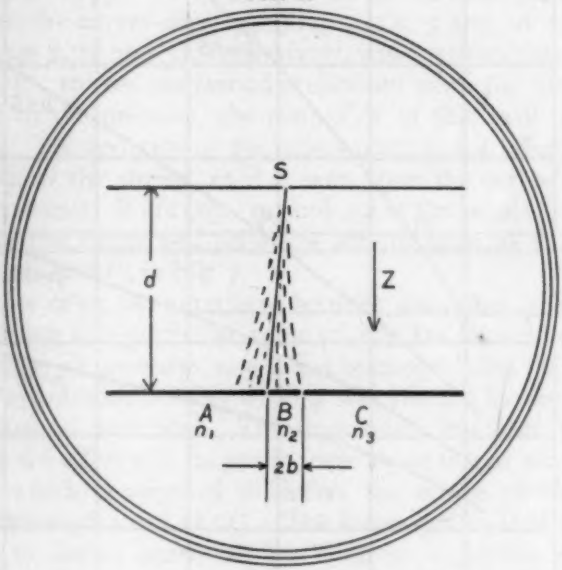
The current in the coils is adjusted so that the centre of the stream follows the line from the centre of the slit to the centre of the gap between the two electrodes A and B. This adjustment is obtained when the charge n_1 received by A is equal to the sum of the charges $(n_2 + n_3)$ received by B and C. Thus the angle of

[12] J. FRANCK, *Deutsch. Phys. Gesellsch.*, Verh. 12, p. 291, 1910, and Verh. 12, p. 613, 1910.

deflection of the stream is $\tan^{-1}(b/d)$, where b is half the distance between the centres of the two gaps on either side of B, and d the distance of the slit from the plane of the receiving electrodes.

In order to find the velocity W from these observations, where the electrons move in a resisting medium, the action of a magnetic force H on the current in a wire may be considered. The current in the wire may be supposed to be due to electrons moving with a velocity W in the direction of the electric force Z along the wire.

FIG. 6.



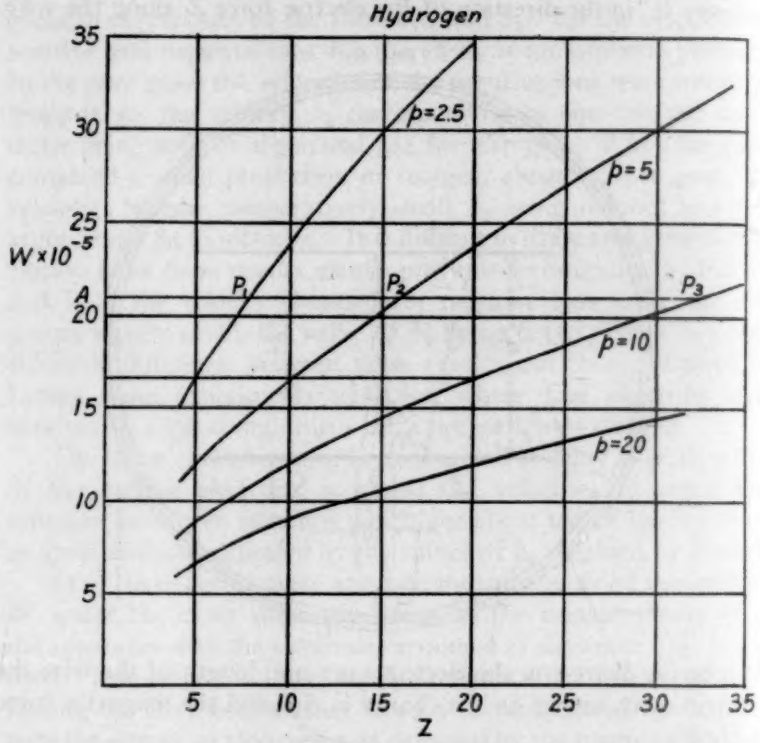
If q be the charge on the electrons per unit length of the wire the electric force acting on the charge is Zq, and the magnetic force is HqW.

There are two similar forces acting on the stream of electrons in the gas, but in this case the stream is not contained within a cylindrical boundary, but is free to move in any direction so that its centre follows the resultant of the two forces which is inclined at the angle $\tan^{-1}(HW/Z)$ to the direction of Z. Hence when the magnetic force H required to give the deflection $\tan^{-1}(b/d)$ is determined the velocity W is obtained from the formula

$$\frac{HW}{Z} = \frac{b}{a} \qquad (14)$$

The effect of the magnetic force H on the direction of motion of the stream of electrons may also be investigated by considering the effect of the curvature of the free paths of the electrons between collisions with molecules.[13] The velocity in the direction perpendicular to the directions of the forces H and Z may be found in a form similar to the expression given in equation (7) for the

FIG. 7.



velocity in the direction of the electric force. The formula obtained for the inclination θ of the direction of motion to the direction of the electric force is $\tan\theta = HW/Z$, which is in agreement with the result obtained from the analogy of the effect of a magnetic force on the current in a wire.

As the centre of the stream may not fall exactly on the centre of the electrode B when the magnetic force is not acting, it is necessary to determine the forces H_1 and H_2, which deflect the stream to the centres of the two air gaps on either side of the

[13] *Proc. Roy. Soc.*, A, **86**, p. 571, 1912.

central electrode, and to substitute $(H_1 + H_2)/2$ for H in formula (10).

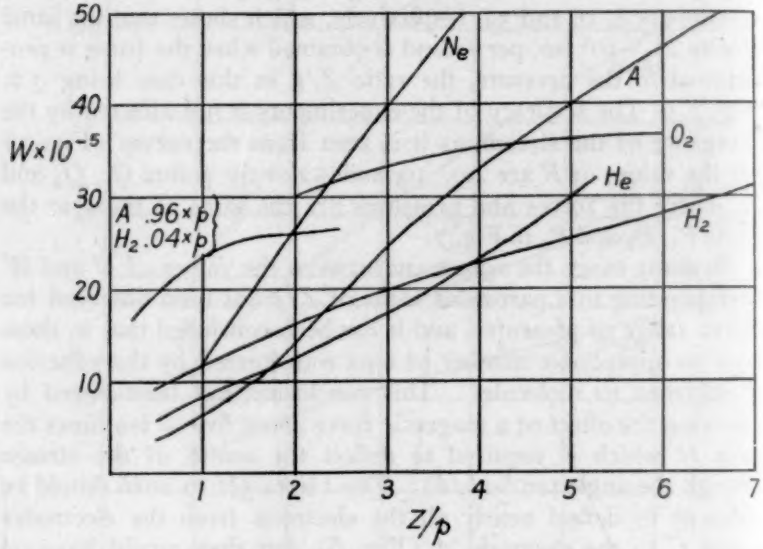
(12) The velocities obtained by this method in hydrogen at various pressures are given by the curves (Fig. 7), where the ordinates are the values of $W \times 10^{-5}$ in centimetres per second, and the abscissæ the electric forces in volts per centimetre. For a given gas pressure, W is not proportional to Z, but it is seen by comparing the curves for different pressures that W is a function of the ratio (Z/p). Thus the abscissæ of the three points P_1, P_2 and P_3 on the curves corresponding to 2.5, 5 and 10 millimetres pressure are 8, 16 and 32, respectively, which shows that the same velocity 21×10^5 cm. per second is obtained when the force is proportional to the pressure, the ratio Z/p in this case being 3.2. (Fig. 7.) The accuracy of the experiments is not affected by the divergence of the stream, as it is seen from the curves (Fig. 3) that the values of R are .29, .39, and .52 at the points Q_1, Q_2 and Q_3, where the forces and pressures are the same as those at the points P_1, P_2 and P_3 in Fig. 7.

In many cases, the agreement between the values of U and W corresponding to a particular value of Z/p has been observed for a large range of pressures, and it has been concluded that in these gases no appreciable number of ions was formed by the adhesion of electrons to molecules. This conclusion has been tested by observing the effect of a magnetic force about five or ten times the force H which is required to deflect the centre of the stream through the angle $\tan^{-1}(b/d)$. The forces $5H$ to $10H$ should be sufficient to deflect nearly all the electrons from the electrodes B and C to the electrode A (Fig. 6), but they would have no appreciable effect on ions, as the velocities W for ions are so small. It was found that in cases where U and W remain constant when Z and p are increased in the same proportion, the charge $(n_2 + n_3)$ received by B and C became negligible in comparison with the charge n_1 received by A, when the magnetic field was increased. The gases which satisfied these tests are nitrogen, carbon dioxide, carbon monoxide, argon, neon and helium.

In some gases the presence of ions was detected, and the proportion of ions to electrons depended both on the pressure p and on the ratio Z/p. In an apparatus of fixed dimensions the number of collisions made by an electron with molecules of the gas in traversing a given distance is proportional to the pressure of the gas when the ratio Z/p is constant. Hence if ions were formed

in some of the contacts of electrons with molecules the proportion of the number of ions to the number of electrons in the stream would increase with the pressure. As a result of this action the observed values of U and W corresponding to the larger pressures are less than those corresponding to the smaller pressures when the ratio of Z/p is maintained constant. This effect was observed in oxygen, nitrous oxide and nitric oxide, and it was also found that a part of the charge received by the electrodes B and C was undeflected by a magnetic force sufficiently large to deflect all the

FIG. 8.



electrons. As the number of ions diminishes with the pressure the velocities U and W obtained with the smaller pressures are taken as the velocities of the electrons in these gases. In oxygen and nitric oxide the proportion of ions to electrons in the stream diminishes at the ratio Z/p increases. With nitrous [14] oxide a different result was obtained. In this gas the number of ions in the stream was small when the ratio Z/p was small, but the number increased with the ratio Z/p.

In air the proportion of ions to electrons was much less than in pure oxygen, and when the ratio Z/p was greater than .3 their effect was negligible over a considerable range of pressures.

(13) The velocities of the electrons in any gas may be repre-

[14] M. F. SKINKER and J. V. WHITE, *Phil. Mag.*, **46**, p. 630, 1923.

sented by a curve giving W in terms of Z/p, as in Fig. 8, where the curves for argon, neon, helium, oxygen and hydrogen are shown. A curve is also given for a gas containing 96 per cent. of argon and 4 per cent. of hydrogen, which shows that a large increase in the velocity is obtained when a small quantity of hydrogen is mixed with argon. In pure argon the velocity of agitation of electrons is very large compared with the velocity in hydrogen as shown by the curves (Fig. 5), so that when hydrogen is mixed with argon the velocity of agitation is reduced, which has the direct effect of increasing the velocity W. Also the mean free path in argon is increased when U is diminished, and this effect also contributes to the increase in W. In these experiments it has been found that other impurities also have a similar effect of increasing the velocity in argon.

Another remarkable difference between the two gases is seen from experiments on the effect on the velocities in hydrogen due to the addition of argon. If a small quantity of argon is added to hydrogen the velocity of the electrons is not appreciably changed, in fact, in order to obtain a noticeable change it is necessary to add a large quantity of argon to the hydrogen. To illustrate this point the following experiment may be quoted.[15]

The velocity W in pure hydrogen at 5.1 millimetres pressure was found to be 1.51×10^6 cm. per second under a force of 8.5 volts per centimetre, and when argon at 20 millimetres pressure was added, bringing the total pressure up to 25.1 millimetres, the velocity under the same electric force was found to be 1.4×10^6 cm. per second. The corresponding change in U was very small, so that equation (7) shows directly the effect on the mean free path l. It thus appears that the reduction in the mean free path of an electron due to the addition of a large quantity of argon to hydrogen is comparatively small. An approximate calculation from the results of the above experiment shows that when the velocity of agitation of the electrons is 4×10^7 cm. per second (corresponding to a potential fall of about half a volt) the mean free path of an electron in pure argon is about fifty times as long as the mean free path in pure hydrogen at the same pressure.

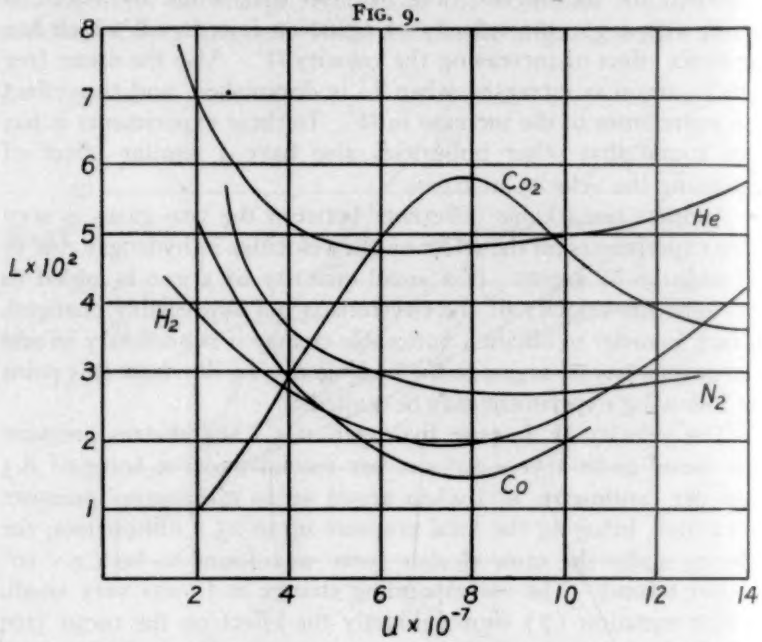
In applying this principle to compare the mean free path of electrons in hydrogen and argon it it necessary to add a sufficient quantity of argon to the hydrogen to make a large change in W. For this purpose, a series of experiments was made with a mixture

[15] J. S. TOWNSEND and V. A. BAILEY, *Phil. Mag.*, **43**, p. 1127, 1922.

where the pressure of the argon was twenty-four times that of the hydrogen. By this means the free paths in argon were obtained for the smaller velocities of agitation of the electrons from $U = 3 \times 10^7$ to $U = 12 \times 10^7$. In pure argon it was found difficult to measure small velocities of agitation as the divergence of the stream of electrons was so large with the smaller electric forces.

(14) From the determinations of the velocities U and W in

FIG. 9.



terms of Z/p, the free paths L are obtained by equation (11), and the values of L corresponding to different values of U may be represented by means of a curve for each gas. Some of the curves thus obtained are given in Figs. 9 and 10 for the range of the velocity U from 2×10^7 to 12×10^7 centimetres per second. Except in the case of carbon dioxide, the free paths increase as the velocity diminishes towards the lower part of this range, which shows that the forces acting on an electron in a collision with a molecule have less effect in deflecting the electron when its velocity is reduced.

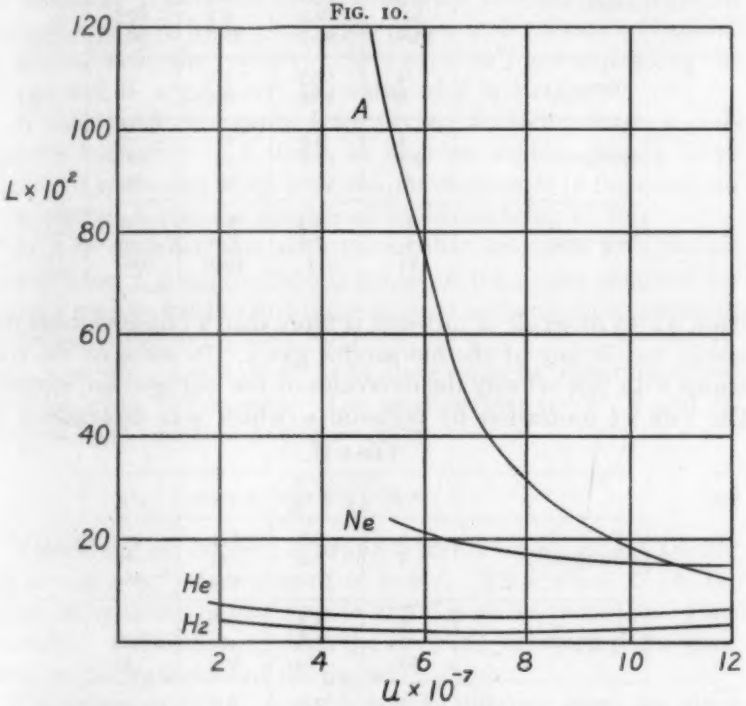
The results obtained with the monatomic gases, as shown by the curves (Fig. 10), are remarkable, as the free paths in these

gases are so long compared with the free paths in other gases, and the greater the atomic weight the longer the free paths.

Thus the mean free path of an electron moving with a velocity of 5×10^7 centimetres per second is 11 millimetres in argon at one millimetre pressure, 2.5 millimetres in neon, .48 millimetre in helium, .32 millimetre in nitrogen and .24 millimetre in hydrogen.

The curve for the free paths in carbon dioxide is of interest,

FIG. 10.



as it shows that the free path has a maximum value corresponding to the velocity 8×10^7 cm. per second, and indicates that the free paths for the other gases may also attain a maximum for velocities of the electrons below 2×10^7. In argon a maximum value of L, 16 millimetres, was obtained with the velocity 3.6×10^7 cm. per second, the free path being 14 millimetres when the velocity was reduced to 2.8×10^7.

(15) The proportion λ of the energy of an electron which is lost in a collision is given by equation (12) ($\lambda = 2.46\, W^2/U^2$). A list of values of λ for the monatomic gases is given in Table I,

which may be compared with the numbers for nitrogen. It is seen from these figures that with the same velocity of impact (12×10^7 cm. per second) the energy of an electron which is lost in a col-

TABLE I.

	Z/p	$U \times 10^{-7}$	$\lambda \times 10^4$	α/p
Argon	.2	12.6	1.6	—
	.52	16.2	1.6	—
	5.0	20.2	9.7	.075
	15.0	20.7	38.6	.30
Neon	.4	12.3	5.8	
	.8	16.0	6.9	
Helium	.5	5.9	23	—
	2.5	12.8	34	.02
	5	15.1	98	.14
Nitrogen	1	5.3	64	—
	5	7.4	330	—
	50	12	4940	.08

lision with a molecule of nitrogen is more than a hundred times the energy lost in any of the monatomic gases. In some of the collisions with this velocity the molecules of the nitrogen are ionized. The rate of ionization by collision α which was determined in

TABLE II.

	Z/p	$U \times 10^{-7}$	$\lambda \times 10^4$	α/p
Nitrogen	1	5.3	6.4	—
	5	7.4	33	—
	50	12	494	.08
Hydrogen	1	3.5	28	—
	5	5.8	46	.006
	50	14	590	.35
Carbon monoxide	1	2.7	100	
	5	5.5	53	
	50	10.6	500	
Carbon dioxide	1	1.4	37	—
	5	3.4	510	—
	50	13.5	510	.15
Oxygen	10	8.1	78	
	50	13.4	550	

previous researches is seen from the last column of figures, where the values of α/p corresponding to the values Z/p are given. The contrast between nitrogen and the monatomic gases is not so marked if the effects with lower velocities are considered. Thus

with a velocity of about 5×10^7 centimetres per second the proportion of the energy of an electron lost in a collision is 2×10^{-4} in helium, and 6×10^{-4} in nitrogen.

(16) With the smaller velocities U the coefficient of elasticity f is nearly equal to the value $(f = 1)$, corresponding to perfect elasticity in the collisions between electrons and molecules. For example, with the velocity $U = 4 \times 10^7$ corresponding to $k = 12$, the values of f obtained from Pidduck's formula are .9998 in nitrogen, .9990 in hydrogen, .9990 in air, and 1.00001 in helium. In argon with the velocity $U = 13.6 \times 10^7$, corresponding to $k = 140$ and $W = 3.4 \times 10^5$, the value of f is 1.000006.

A very small divergence from perfect elasticity causes a considerable reduction in k, since an electron collides with a large number of molecules when it moves one centimetre in the direction of the electric force; the number of collisions being U/W_1.

If it be assumed that with the smaller velocities of agitation the coefficient f is approximately constant, the values obtained for W and k may be used to find the energy of agitation of electrons in dynamical equilibrium with molecules of a gas when no electric force is acting. Let $k'M\,\Omega^2/2$ be the energy of agitation of the electrons when W' is zero. The relation between k', k and W obtained from equation (13) reduces to

$$\frac{k}{k'} = \frac{W^2}{\Omega^2} + 1 \qquad (15)$$

Taking the experiment in argon where $k = 140$, $W = 3.4 \times 10^5$ and $\Omega = 42 \times 10^3$, the value of k' is 2.1. Thus when $Z = 0$, the energy of agitation of electrons in argon is about twice that of the molecules. In helium and neon the electrons have nearly the same energy as the molecules of the gas when $Z = 0$.

The values of W, U, L and λ for the different gases are given in the following tables: [16]

[16] The velocities U and W, which have been determined experimentally, are given in the following papers:

Air.—J. S. TOWNSEND and H. T. TIZARD, *Proc. Roy. Soc.*, A, **88**, p. 336, 1913.

Oxygen, Hydrogen, Nitrogen, Argon, Helium.—J. S. TOWNSEND and V. A. BAILEY, *Phil. Mag.*, **42**, p. 873, 1921; **43**, p. 593 and p. 1127, 1922; **44**, p. 1033, 1922; **46**, p. 657, 1923.

Carbon Dioxide.—M. F. SKINKER, *Phil. Mag.*, **44**, p. 994, 1922.

Carbon Monoxide, Nitrous Oxide and Nitric Oxide.—M. F. SKINKER and J. V. WHITE, *Phil. Mag.*, **46**, p. 630, 1923.

Neon.—V. A. BAILEY, *Phil. Mag.*, **47**, p. 379, 1924.

Air.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
100	160	145	270	2.72	860
50	102	116	173	2.78	550
20	57	87	90	2.71	260
10	46	78	52	2.82	113
5	38	71	30	2.96	43
2	22	54	17.5	3.28	26
1	11	38	12.5	3.30	26
0.5	5.7	27	9	3.37	26

Nitrogen.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
60	126	129	193	2.89	550
40	89	108	146	2.75	448
20	59.5	88.5	86	2.66	234
10	48.5	80	48.5	2.69	90
5	41.3	74	27	2.77	33
2	30.5	63.5	13.1	2.88	10.3
1	21.5	53 5	8.7	3.20	6.5
0.5	13.0	41.4	6.2	3.55	5.5
0.25	7.5	31.5	5.15	4.50	6.5

Oxygen.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{-4}$
50	136	134	201	3.74	550
20	70	96	86	2.89	197
10	50	81.2	46	2.58	79
6	45	77.1	36	3.22	53
2	22.5	54.5	30	5.6	74

Hydrogen.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
50	148	140	217	4.2	590
40	130	131	160	3.67	366
20	78	101.5	70	2.5	117
10	44	76.2	38	2.05	62
5	26.4	59	25.5	2.14	46
2	15	43	16	2.39	34
1	9.3	35	11.9	2.86	28.5
0.5	5.4	26.5	9.0	3.25	29
0.25	3.1	20.2	6.5	3.64	26

Argon.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{5}$
15	324	207	82	7.9	38.6
10	324	207	65	9.4	24.3
5	310	202	40	11.3	9.7
1.25	320	206	7.7	8.9	3.45
.95	280	193	6.0	8.5	2.38
.71	240	178	4.85	8.5	1.82
.525	200	163	4.15	9.0	1.60
.355	160	145	3.6	10.3	1.52
.195	120	126	3.25	14.7	1.64

Mixture of Hydrogen and Argon.

Hydrogen at partial pressure p, argon at partial pressure $24 \times p$.

L_m, the mean free path in the mixture, containing hydrogen at one millimetre pressure and argon at 24 millimetres.

L_h, the mean free path in pure hydrogen at one millimetre pressure, corresponding to velocity of agitation U.

L_a, the mean free path in pure argon at one millimetre pressure, corresponding to velocity of agitation U.

The free path L_a is obtained from L_m and L_h by the formula

$$\frac{1}{L_m} = \frac{24}{L_a} + \frac{1}{L_h}$$

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L_m \times 100$	$L_h \times 100$	$L_a \times 100$
64.8	140	136	26.5	.39	4.00	10.4
42.4	100	115	25.4	.48	2.98	13.6
26	70	96	23.5	.61	2.30	20.0
15.6	50	81.5	20.8	.76	2.00	29.5
6.55	30	63	16.7	1.12	1.99	61.5
2.25	16	46	12.2	1.75	2.49	138
1.28	10	36.4	10.0	1.99	2.88	161
1.0	8	32.6	9.1	2.08	3.06	154
.75	6	28.2	7.9	2.09	3.30	138

Neon (Containing 1 per Cent. of Helium).

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{5}$
6	316	204	74.5	17.7	330
4	308	202	53.0	18.7	170
2	275	191	27.0	18.6	49
1.2	235	176	15.0	15.4	18
.8	194	160	9.65	13.5	9.0
.6	158	145	7.95	13.4	7.4
.4	114	123	6.35	13.7	6.6
.2	62	90.5	5.0	15.8	7.5
.1	34.5	67.5	3.95	18.7	8.4
.06	23.5	55.8	3.3	21.5	8.6

Helium.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
5	172	151	30.2	6.4	9.8
4	152	142	23.5	5.85	6.75
3	137	135	17.5	5.5	4.15
2	105	118	12.7	5.25	2.85
1	53	84	8.25	4.85	2.4
.5	27	59.6	5.74	4.8	2.3
.2	11.3	38.7	3.93	5.3	2.5
.1	6.2	28.7	2.96	5.95	2.6
.05	3.68	21.2	2.14	6.6	2.3
.02	2.12	16.8	1.33	7.8	1.56
.013	1.77	15.3	1.11	9.14	1.30

Carbon Dioxide.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
50	139	136	195	3.67	506
20	75	99.5	138	4.76	472
10	47	79	108	5.91	460
5	9	34.5	50	2.39	516
2	1.8	15.4	11.8	0.63	144
1	1.5	14.1	5.5	.54	37.4
0.5	1.3	13.1	2.5	.45	8.95
0.25	1.2	12.6	1.2	.42	2.34

Carbon Monoxide.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
50	86	107	152	2.24	500
20	42	74.5	57	1.47	144
10	34	67	38	1.76	79
5	22.7	54.8	28.5	2.16	52.8
2	11.4	38.2	23.2	3.04	89.1
1	7.0	30.5	18.0	3.8	86
0.5	4.8	25.2	13.0	4.5	65.5
0.25	3.2	20.6	9.0	5.14	46.9

Nitric Oxide.

Z/p	k	$U \times 10^{-6}$	$W \times 10^{-5}$	$L \times 100$	$\lambda \times 10^{4}$
10	21.7	53.6	23.6	.88	47.7
5	13.3	41.9	14.5	.84	29.4
4	12.2	40.2	13.9	.97	29.5
3	11.0	38.2	12.5	1.10	26.4
2	9.3	35.1	10.0	1.21	20.0
1	7.0	30.4	6.5	1.37	11.2

A DERIVATION OF SAHA'S EQUATION FOR TEMPERATURE IONIZATION.

BY

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THE derivation here given, when written out, seems so obvious that an apology seems almost necessary for presenting it. The writer has, however, been conscious of a feeling of diffusiveness of principles in the proofs which he has seen, and has found this feeling shared by others. For this reason the following derivation may be of some value.

The entropy S of a monatomic gas of N atoms is [1]

$$S = Rn\left(\frac{5}{2}\log T - \log p + \frac{3}{2}\log m + S_0\right) \qquad (1)$$

where R is the gas constant, p the pressure, m the atomic weight and S_0 a constant. RS_0 is the entropy constant as ordinarily understood.

For a mixture of gases

$$S = \Sigma Rn_\nu\left(\frac{5}{2}\log T - \log p_\nu + \frac{3}{2}\log m_\nu + S_0\right)$$

where ν designates a particular gas, and p_ν is the partial pressure for that gas.

In the case under consideration, that of a monatomic gas dissociating into positive ions and electrons, the summation com-

[1] See, for example, Planck's "Vorlesungen über die Theorie der Wärmestrahlung," fourth edition, p. 211, where

$$S = Rn\log\frac{Ve}{nh^3}\,[2\pi emRT]^{\frac{3}{2}},$$

h being Planck's constant, and e the Naperian base. On writing $pV = nRT$, we obtain

$$S = Rn\left[\frac{5}{2}\log T - \log p + \frac{3}{2}\log m + \log\frac{(2\pi)^{\frac{3}{2}}(Re)^{\frac{5}{2}}}{h^3}\right]$$

which gives S_0 in terms of R and h.

This expression is obtained on the assumption that h^3 represents the volume of the state element in the six-dimensional state space for a molecule. Other assumptions as regards the volume of the state element merely modify the expression for S in the last term, *i.e.*, the constant term.

prises these terms. If δn represents an arbitrary change in the number of neutral atoms, δn_+ and δn_- the corresponding changes for the positive ions and electrons, the corresponding change δS in the entropy is

$$\delta S = (\delta n + \delta n_+ + \delta n_-)\left(\frac{5}{2}\log T + S_0\right)R + \frac{3}{2}R(\delta n \log m + \delta n_+ \log m_+ + \delta n_- \log m_-) - R(\delta n \log p + \delta n_+ \log p_+ + \delta n_- \log p_-) - R\left(n\frac{\delta p}{p} + n_+\frac{\delta p_+}{p_+} + n_-\frac{\delta p_-}{p_-}\right) \quad (2)$$

Now

$$\delta n_+ = \delta n_- = -\delta n \quad (3)$$

also

$$\frac{n}{p} = \frac{n_+}{p_+} = \frac{n_-}{p_-} = \frac{n + n_+ + n_-}{p + p_+ + p_-} \quad (4)$$

so that

$$n\frac{\delta p}{p} + n_+\frac{\delta p_+}{p_+} + n_-\frac{\delta p_-}{p_-} = \frac{n}{p}(\delta p + \delta p_+ + \delta p_-) \quad (5)$$

If the change occurs at constant pressure

$$\delta p + \delta p_+ + \delta p_- = 0 \quad (6)$$

so that, using (3), equation (2) becomes

$$\delta S = -\delta n\left(\frac{5}{2}\log T + S_0\right)R - \frac{3}{2}R\delta n \log\frac{m_+ m_-}{m} + R\delta n \log\frac{p_+ p_-}{p} \quad (7)$$

But, if Q_p is the heat absorbed at constant pressure on account of the ionization of the $-\delta n$ atoms [2]

$$\delta S = \frac{Qp}{T} = \frac{-U\varepsilon\delta n + \frac{3}{2}RT(\delta n + \delta n_+ + \delta n_-) + P\delta V}{T} \quad (8)$$

where U is the ionization potential, ε the electronic charge, P the *total* pressure, and δV the change of volume associated with the ionization of the $-\delta n$ atoms.

The $\frac{3}{2}RT(\delta n + \delta n_+ + \delta n_-)$ refers to the thermal energy which has to be given to raise the increased number of entities $(\delta n + \delta n_+ + \delta n_-)$ to the temperature T. Observing that

[2] The minus sign occurs because δn refers to the increase in the number of neutral atoms.

$$V = (n + n_+ + n_-)\frac{RT}{P}$$

$$V + \delta V = (n + \delta n + n_+ + \delta n_+ + n_- + \delta n_-)\frac{RT}{P}$$

so that

$$\delta V = \frac{(\delta n + \delta n_+ + \delta n_-)}{P} RT \tag{9}$$

and using (3)

$$P\delta V = -RT\delta n \tag{10}$$

and again using (3), (8) becomes

$$\delta S = \frac{-U\varepsilon\delta n - \frac{3}{2}RT\delta n - \delta nRT}{T} \tag{11}$$

so that, equating (7) and (11), and noting that $\frac{m_+}{m} = 1$, approximately, if the negative ions are electrons, we have

$$\log \frac{p_+ p_-}{p} = -\frac{U\varepsilon}{RT} + \frac{5}{2}\log T + \frac{3}{2}\log m_- - \frac{5}{2} + S_0 \tag{12}$$

If x is the fraction of the atoms ionized,

$$x = \frac{n_+}{n + n_+} = \frac{p_+}{p + p_+}, \; p + p_+ + p_- = P; \; p_+ = p_-$$

from which, it follows that

$$\frac{p_+ p_-}{p} = \frac{x^2}{1 - x^2} P$$

and there results immediately Saha's equation

$$\log \frac{x^2}{1 - x^2} P = -\frac{U\varepsilon}{RT} + \frac{5}{2}\log T + \frac{3}{2}\log m_- - \frac{5}{2} + S_0 \tag{13}$$

If we imagine the arbitrary change δn to take place at constant volume, we of course arrive at the same result. For, since

$$\frac{\delta p}{p} = \frac{\delta n}{n}; \; \frac{\delta p_+}{p_+} = \frac{\delta n_+}{n_+}, \; \frac{\delta p_-}{p_-} = \frac{\delta n_-}{n_-},$$

we have for the last term of (2)

$$R\left(n\frac{\delta p}{p} + n_+\frac{\delta p_+}{p_+} + n_-\frac{\delta p_-}{p_-}\right) = R(\delta n + \delta n_+ + \delta n_-) = -R\delta n. \tag{14}$$

Hence (7) becomes replaced by

$$\delta S = -\delta n\left(\frac{5}{2}\log T + S_0\right) R - \frac{3}{2}R\delta n \log \frac{m_+ m_-}{m} + R\delta n \log \frac{p_+ p_-}{p} + R\delta n \tag{15}$$

But, in this case, the term $\frac{P\delta V}{T}$ must be omitted from (8); and, by (10) this amounts to subtracting $-R\delta n$ from (11), so that (11) becomes

$$\delta S = \frac{-U\Sigma\delta n - \frac{3}{2}RT\delta n}{T} \qquad (16)$$

The result of equating (15) and (16) is the same as that of equating (7) and (11), and leads immediately to (12) and thence to Saha's equation (13).

The Transmutation of Mercury into Gold. H. NAGAOKA. (*Naturwissenschaften*, July 31, 1925.)—From his investigations of the spectral lines of gold and mercury this Japanese physicist was led to regard as promising an attempt to transmute the base into the noble metal. He accordingly sent the discharge of an induction coil of maximum spark length of 120 cm. from a tungsten point to a body of mercury covered with oil. Before the experiment the mercury was chemically purified, and in addition was distilled at least twice in a vacuum. Under the action of the discharge the mercury forms a sort of mixture with the charred oil, and after two or more hours it furnishes a reaction for gold. When, after preliminary treatment, the residue was extracted with aqua regia, unmistakable traces of gold were found. Parallel experiments conducted in the absence of the discharge produced no such evidences of gold. Further, in some cases of the experiment small particles of metallic gold were obtained.

The disruption of the mercury atom takes place only in fields of several million volts per cm. A fuller account of these highly interesting investigations will appear in the *Japanese Journal of Physics*.

The International Astronomical Union met in July, 1925, at Cambridge, England. In connection with this the University conferred the honorary degree of doctor of science upon five of the members of the Union. Of these, two, President Campbell and Doctor Schlesinger, were from the United States; one, W. de Sitter, from Holland; one, B. Baillaud, from France, and the fifth was the author of the article reviewed above. In presenting him the Public Orator, T. R. Glover, said, "Light was once more sought from the East, and a Japanese astronomer came well skilled to track the footsteps of the fugitive atom. A shrewd and able investigator, he had quite recently invited the men of science to decide whether in point of fact he really had made gold out of humbler atoms by transmutation." G. F. S.

THEORY OF THE RANGE OF ALPHA-PARTICLES.*

BY

LEONARD LOEB, Ph.D., and EDWARD CONDON.

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THE theory of the range of α-particles has been studied by J. J. Thomson,[1] Darwin,[2] and Bohr.[3] In 1922 Henderson[4] attempted to apply quantum conditions for ionization and resonance to the limits of integration of the classical equation for loss of energy of the α-particle to the electrons in the atom. He believed that he had obtained a fairly close agreement between his theory and experiment. Later Fowler,[5] on the basis of a more accurate knowledge of ionization and resonance potentials, showed that this agreement was not as good as Henderson believed it to be. Lately Bates[6] has compared more accurate computations of the ranges with actual measurements made on the stopping powers in various inert gases. He found that in general the calculated stopping power was from 0.6 to 0.8 of the observed stopping power in the gases He, Ne, Ar, Kr and Xe.

In surveying these papers it appeared to the writers that in the use of the theory as derived by Henderson, a tacit assumption had been made which was not necessarily justified. This assumption, if incorrect, would serve to make the calculated stopping powers greater than they should be. It is the purpose of this paper to indicate the nature of the error introduced into the computations and to estimate its magnitude for the case of He, the one gas where this is possible. It was found that for this case the error introduced was such as to make the calculated stopping powers appear markedly greater than they should be. Further considerations indicate that for the other gases the error introduced is of the same order of magnitude as for He. In all cases they therefore serve to increase the already existing discrepancy between calculated and observed stopping power. Inasmuch as the considerations which lead to these conclusions may be of use to other workers in this field, the writers venture to publish a short account of the calculations.

Classical dynamics shows that for a free electron of charge e

* Communicated by Dr. W. F. G. Swann, Associate Editor.

and mass m initially at rest, an α-particle of charge E, mass M and velocity V will impart an energy

$$Q(p) = \frac{2E^2e^2}{mV^2(p^2 + a^2)}$$

to the electron if the perpendicular from the initial position of the electron to the path of the α-particle is p. The quantity a is given by

$$a = \frac{Ee(M + m)}{MmV^2}$$

which may be written as $\frac{Ee}{MV^2}$ without serious error. If one draw about the path of the α-particle two cylinders of radii p and $p + dp$ and of length Δx, any electron that lies in the volume $2\pi p dp \Delta x$ will be given an energy $Q(p)$. On the concept of quantized energy exchanges an electron whose ionization potential is T_s in equivalent volts will be ionized if $Q(p)$ in equivalent volts be equal to or greater than T_s. Now Henderson, as well as Bates and Fowler, makes the assumption that if there are N atoms per cm.^{3} having n_s electrons whose ionization potential is T_s, the loss of energy per cm. of path will be:

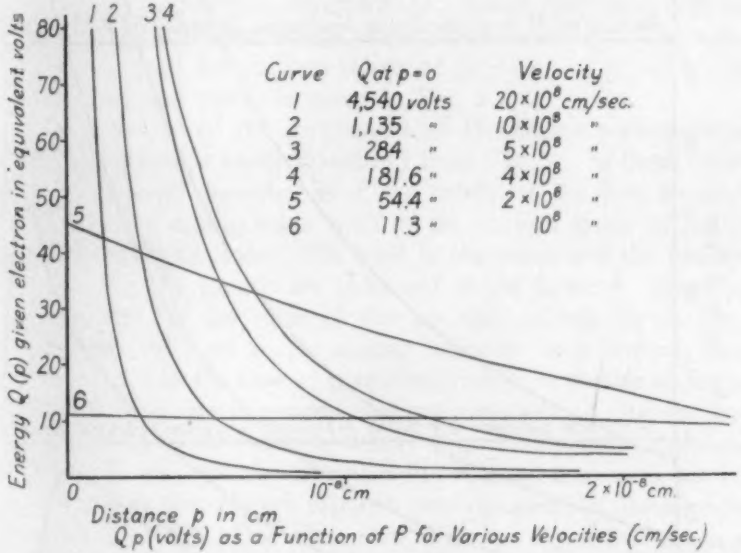
$$\pi N n_s \int_0^{p_s} Q(p)\, d(p^2) = \frac{A n_s}{V^2} \log \frac{4W}{T_s}$$

Here p_s is the value of p for which $Q(p)$ equals T_s, A is given by $\frac{2\pi NE^2e^2}{m}$, and W is the kinetic energy of an electron having a velocity V. This means that *all of the n_s electrons* having an ionizing potential T_s that lie within p_s of the path of the α-particle will be ionized and will receive an energy T_s or more.

The use of the factor n_s above is perfectly valid, provided the n_s electrons can be regarded as independently distributed at random along the path of the α-particle. Now actually the n_s electrons are related, and are part of a single unit: The atom. They are not completely independent and they are not necessarily distributed at random relative to the cylinders of radius p_s. If p_s is sufficiently small compared with the dimensions of the atom, so that the α-particles can act selectively on only one electron of each atom, the random distribution exists and the use of the factor n_s is justified. If, however, p_s becomes comparable with the radius of the atom, more than one electron may be removed from the same atom or even the same electron group in the atom.

and multiplying the integral of $Q(p)$ by n_s permits each of these n_s electrons to be removed from that atom with an energy corresponding to T_s or more. It is, however, well known that to remove two electrons from the same level requires more than twice the energy for one electron and to remove two electrons of different groups in the same atom takes more energy than to remove each one singly. Thus an error is introduced which may be very great in case n_s were 8 (as it is in some atoms), and T_s

FIG. 1.

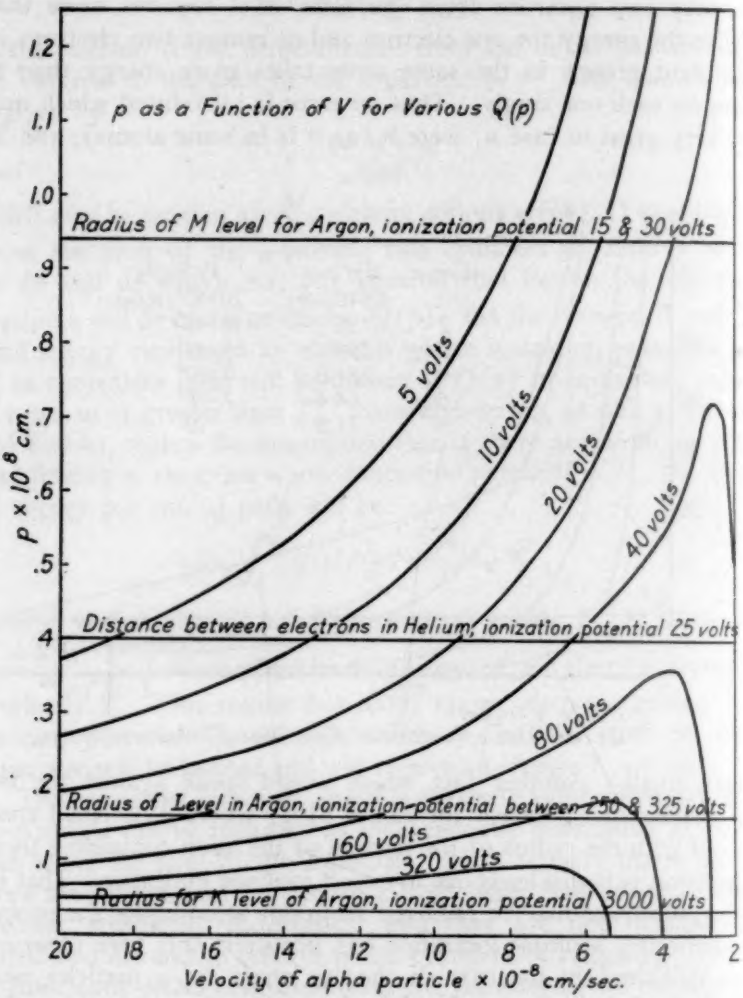


Q p (volts) as a Function of P for Various Velocities (cm/sec.)

were small. Another fact which would speak against the use of the factor n_s, except for values of T_s where p_s is small compared with the radius of the portion of the atom containing these electrons, is that it leads one to expect multiple ionization. That is, all n_s electrons may be removed from one atom under the proper conditions. Multiple ionization has, however, only been observed by Millikan [7] in the case of the He atom for α-particles near the end of their range. No other atoms seem to show even this double ionization. Thus if a study should show that p_s is of the order of magnitude of the orbits of the electrons of energy T_s, it is evident that experimental as well as theoretical considerations will indicate that the use of the factor n_s is wrong. In an

extreme case it is conceivable that only one electron could be removed from each level. In this case the n_s becomes 1. Practi-

FIG. 2.



cally, as will be seen, the considerations lead one to a value intermediate between the two extremes.

In Fig. 1 are plotted the curves for $Q(p)$ in volts as a function of p for various velocities V of the α-particle. From these it

is seen that p to give 20 volts to an electron for the fastest α-particles is $.19 \times 10^{-8}$ cm., while to give one electron 80 volts p must be $.1 \times 10^{-8}$ cm. Near the end of the range the α-particle can give 20 volts at $p = .75 \times 10^{-8}$ cm. and 80 volts at $.32 \times 10^{-8}$ cm. At still lower speeds the α-particle can give 10 volts (but no more) over distances of 2×10^{-8} cm. Hence the rate of ionization is very rapid in this region. However, only the outer electrons of the atoms can be removed. Also p is so large that while several of these lie within a distance p, only one of them can receive enough energy to remove it, *i.e.*, double ionization does not occur. Thus it is seen that p varies widely with V. The variation of p with V for various values of $Q(p)$ (*e.g.*, 5, 10, 20, 40, 80, 160 and 320 volts) is given in Fig. 2.

The question of the correctness of Henderson's assumptions can be qualitatively easily visualized from Fig. 2. In these curves the approximate diameter d of the orbits of the two electrons in He and the approximate radii of the various levels in Ar are plotted as straight lines. The level in the atom and the ionizing potential for the groups are indicated in the figure.† Thus one sees that for He the value of the 20- and 40-volt curves for p exceeds the value of d, the atomic diameter in a limited range of velocities. In the case of potassium, which is similar to argon, the values of p_s are in general distinctly less than d_s for the level corresponding to the ionizing energy T_s. This possibly furnishes a picture of why double ionization in He may occur near the end of the range. For if each electron receives 40 volts, double ionization is possible. One then sees that for a narrow range of velocities the value of p for which $Q(p)$ equals 40 volts exceeds the average distance between the electrons in He. In argon p is less than the radius of the atom and the probability becomes less; argon, however, has many more electrons in each of these levels.

Thus it is seen that in He, and probably also in Ar, the assump-

† The choice of 4×10^{-9} cm. for this radius (*i.e.*, for the average distance between the two electrons in He) came partly from estimates based on the Bohr theory, and partly because of its fitting Millikan's data on double ionization. As the values of the radii of the shells for argon were not known, values for these in K, which differs but little from argon, were taken from computations of Messrs. Brackett and Snoddy. The value of this radius as given could be taken roughly as the average distance between the electrons in the shells which are used in the later considerations.

tions, implied by Henderson, that the radii of the cylinders p_s corresponding to the Sth group of electrons in the atom are so small compared with the spatial distribution that the chances of multiple ionization in a single atom will be negligible, are incorrect. It remains now to make an estimate of the errors resulting from this assumption. For the case of atoms other than He the difficulties of solution are almost insuperable. Even for He only an approximate computation can be made. In what follows the computations will be carried through for He.

To calculate the loss of energy in He the ionization must be divided into two distinct processes, that is, double ionization processes and single ionization processes. For it is well known that while one electron is removed at about 25 volts, two electrons from the same atom require not 50 volts but 80 volts. Thus if they were removed separately from the same atom the first electron would require a p corresponding to a $Q(p)$ of 25 volts, while the second electron would require a p corresponding to a $Q(p)$ of 55 volts. To determine the average energy loss for double ionization it would be simpler to assume that the electrons were liberated simultaneously by the passage of the α-particle. Then each one may receive any energy from 25 to 55 volts, such that the sum of the energies given is above the 80 volts for double ionization. A rough average might be taken in saying that double ionization will result if each of the two electrons lies in a cylinder of radius p_1, corresponding to a $Q(p)$ of 40 volts. In any case the error involved in small changes of p for double ionization would not change the result materially. This can be seen from the curves of Fig. 5 to be discussed later. For single ionization either one or both electrons must lie within a cylinder of radius p_2 corresponding to a $Q(p)$ of 25 volts.‡

Having now given values to p_1 and p_2 for double and single ionization, one may proceed to discuss the numbers of ions formed by these two processes, and hence the energy loss.

One may consider one of the electrons as situated at a point. Then on the average the other electron will be placed at any point

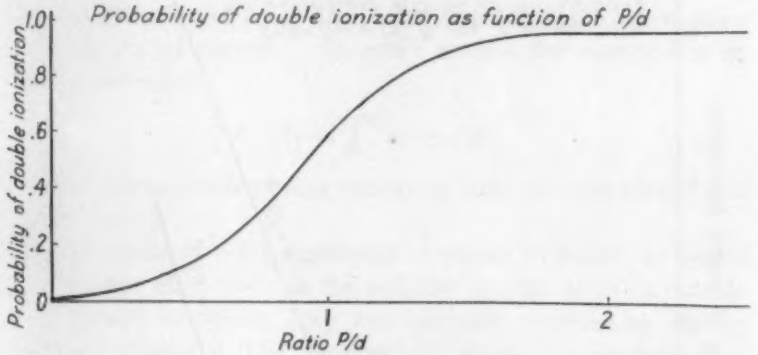
‡ In the actual computation this value p_2 and hence of $Q(p_2)$ for single ionization will be taken as 19.45 volts for He (*i.e.*, its first resonance potential). Fowler has shown that owing to energy losses to resonance the range is shortened to approximately the same extent as if the ionization had set in at their resonance potential.

on the surface of a sphere of radius d drawn about the first electron.

If it requires a value of p equal to p_1 to remove both the electrons, the chance of double ionization can be estimated by finding the fraction of the surface of the sphere of radius d intercepted by the cylinder of radius p_1 drawn about the path of the α-particle relative to the whole surface of the sphere.

The first electron may, however, be placed at any point in

FIG. 3.



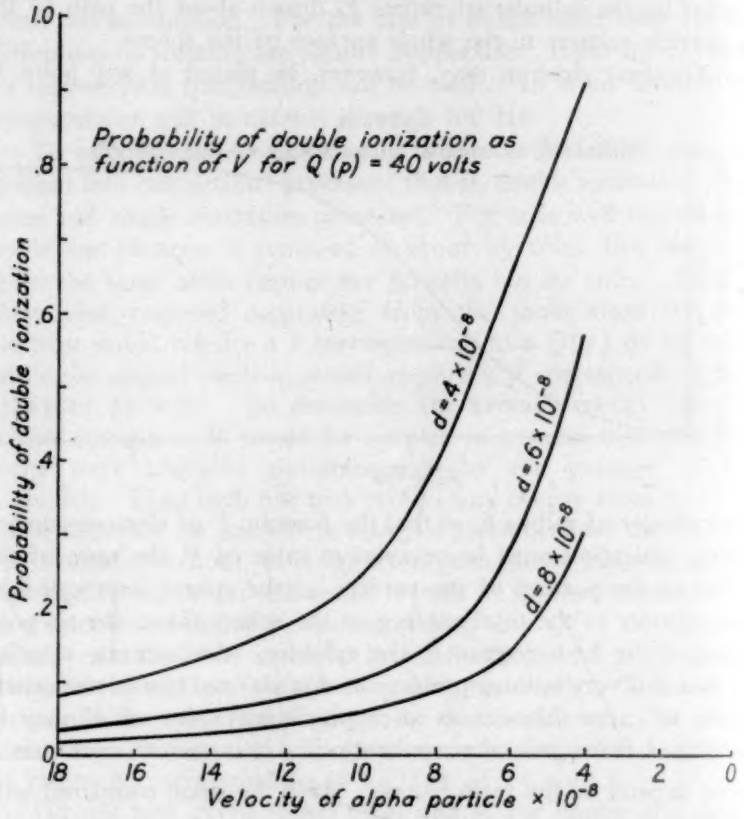
the cylinder of radius p_1 so that the fraction P of electrons undergoing ionization must be an average value of P, the ratio of the areas of the portion of the surface of the sphere intercepted by the cylinder to the total surface of the sphere taken for all positions of the first electron in the cylinder. An accurate solution of this is a very tedious problem, and it was not considered worth while to carry this out as an approximate value of P may be estimated from general considerations. It is seen at once that P must depend on the ratio $\frac{p_1}{d}$ only. If p_1 be small compared with d, then it readily follows that

$$P = \frac{1}{2}\left(\frac{p_1}{d}\right)^2.$$

For values of $p_1 \gg d$ the value of P rapidly approaches unity. A rough calculation of this sort gives for the form of the relation between P and the ratio $\frac{p_1}{d}$ the curve shown in Fig. 3. From this rough curve assigning the p_1, the value assumed for double ionization in He, curves for P were computed for three values of d

at different values of the velocity. These are shown in Fig. 4. It may be seen at once that for $d = 4 \times 10^{-9}$ cm. the values of P give about the same relative values for the fraction of the double ionizations occurring in He at different parts of the range that

FIG. 4.



were estimated by Fowler from Millikan and Wilkins' experiments. Thus at the centre of the range Fowler estimated that 30 per cent. of the ionizations result in double ionization and near the end of the range the estimated value is 75 per cent.||

|| S. Rosseland,[9] using a theory of ionization by α-particles due to J. J. Thomson, made a similar analysis. The results differ from the above in that he used the percentage of double catches directly instead of allowing for secondary ionization as was done by Fowler.

Having thus an approximate value of P as a function of V and a value of d for He, one can proceed to calculate the loss of energy by the α-particles in He. There are N atoms per c.c. present, containing 2 N electrons. Thus the volume of a cylindrical shell of radii p and $p+dp$ and Δx cm. long contains $2\pi Npdp\Delta x$ atoms and hence $4\pi pdp\Delta x$ electrons. In other words, the cylinder $\pi p_1^2\Delta x$ of radius p_1 (corresponding to $Q(p_1)$ equals 40 volts) contains $2\pi Np_1^2\Delta x$ electrons. Of these the fraction P represents *pairs* of electrons that come from the same atom. The number of *pairs* is $\frac{1}{2}P2N\pi p_1^2\Delta x$. But each electron of these pairs receives $Q(p)$ of energy. In other words, the energy lost to double ionization is

$$\frac{1}{2}P\cdot 2N\pi\Delta x\int_0^{p_1}2Q(p)\,d(p^2),$$

for each electron inside p_1 may receive at least 40 volts energy and probably more.§

The fraction $(1-P)$ represents electrons of atoms so placed that while one electron is in the cylinder p_1, the other is outside of it. Hence for these only one electron receives an energy $Q(p_1)$ or greater. The number of atoms so concerned is $\frac{1}{2}(1-P)2N\pi p_1^2\Delta x$ and only the electron within p_1 of these receives the energy of $Q(p_1)$ or more. So that the energy lost in this process is

$$\frac{1}{2}(1-P)2N\pi\int_0^{p_1}Q(p)\,d(p^2)\,\Delta x.$$

Of the electrons between p_1 and p_2, which receive only enough energy to remove one electron with from 20 to 40 volts of energy, a fraction P' will be companions of the electrons in the inner cylinder. The energy given to these has already been computed. The fraction $(1-P')$ will represent electron pairs with one electron in the single ionization region (*i.e.*, between p_1 and p_2) and the other either also in this region or outside of it. Of these, P'' is the fraction where both lie between p_1 and p_2, and $(1-P'')$ where one of them lies outside of it. The losses due to these

§ Strictly P is a function of p and of V, the velocity of the α-particle. However, the P used here is the value taken from the curve of Fig. 3, which represents an average value, and each P used is averaged over the whole range of velocities involved.

electrons are then given on the basis of reasoning similar to the above, as

$$\frac{1}{2}(1-P')P''2N\pi\int_{p_1}^{p_2}Q(p)\,d(p^2)\,\Delta x$$

and

$$(1-P')(1-P'')2N\pi\int_{p_1}^{p_2}Q(p)\,d(p^2)\,\Delta x.$$

In the first case only one electron of the pair *may* be ionized, while in the second case there is *only one present* which will be ionized.

Thus the total loss of energy in unit length is given by

$$\begin{aligned}\frac{dT}{dx} &= P2N\pi\int_0^{p_1}Q(p)\,d(p^2)+\frac{1}{2}(1-P)2N\pi\int_0^{p_1}Q(p)\,d(p^2)\\ &\quad+\frac{1}{2}(1-P')P''2N\pi\int_{p_1}^{p_2}Q(p)\,d(p^2)\\ &\quad+(1-P')(1-P'')2N\pi\int_{p_1}^{p_2}Q(p)\,d(p^2)\\ &=2\pi N\left\{\frac{1}{2}(P+1)\int_0^{p_1}Q(p)\,d(p^2)+(1-P')\left(1-\frac{1}{2}P''\right)\int_{p_1}^{p_2}Q(p)\,d(p^2)\right\}.\end{aligned}$$

If $P=1$ and $P'=P''=0$, one obtains Henderson's expression

$$\frac{dT}{dx}=2\pi N\int_0^{p_2}Q(p)\,d(p^2)$$

The evaluation of the average P for different velocities was made on the basis of the curve of Fig. 4. P' was set as nearly equal to P. Although P' would be perhaps slightly greater than P, it does not deviate materially from it. P'' was difficult to compute, but was taken roughly as the ratio

$$P''=\frac{P_{p_2}-P_{p_1}}{1-P_{p_1}}$$

Where P_{p_2} is the value of the average fraction of the surface intercepted by the cylinder p_2 on the spherical surface of radius d, and P_{p_1} was the average fraction intercepted on the surface of the sphere by the cylinder p_1, that is P_{p_1} is the quantity P used before. The values taken for P, P' and P'' are then, $P=.34$, $P'=.34$, and $P''=.37$.

This leads to an expression for

$$\frac{dT}{dx}=2N\pi\left\{0.67\int_0^{p_1}Q(p)\,d(p^2)+.56\int_{p_1}^{p_2}Q(p)\,d(p^2)\right\}.$$

Remembering that

$$Q(p)=\frac{8e^4}{mV^2(p^2+a^2)}$$

one obtains

$$\frac{dT}{dx} = \frac{16\pi e^4 N}{mV^2} \times .67 \log \frac{4W}{Q_1^{.16} Q_2^{.84}}$$

or

$$-MV\frac{dV}{dx} = \left(\frac{8\pi e^4 N}{mV^2}\right) \times 1.34 \log \frac{4W}{Q_1^{.16} Q_2^{.84}}$$

Henderson's equation takes the form under similar conditions

$$-MV\frac{dV}{dx} = \left(\frac{8\pi e^4 N}{mV^2}\right) \times 2 \log \frac{4W}{Q_2}.$$

The *exact integration* of these expressions for the range of velocities used gives the following results:

(1) *Experimental range* in He according to Bates = 23.6 cm.
(2) Computed range in He according to Henderson = 33.0 cm.
(3) Computed range in He on corrected theory = 58 cm.

The latter value, while only rough, indicates the order of magnitude of the correction to Henderson's theory.

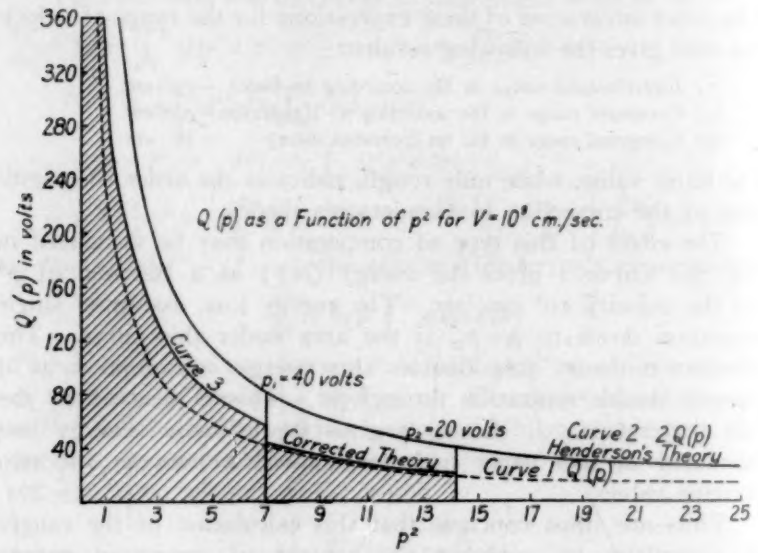
The effect of this type of computation may be visualized in Fig. 5. Curve 1 gives the energy $Q(p)$ as a function of p^2 for the velocity 10^9 cm./sec. The energy loss, assuming single ionization down to $p = p_2$, is the area under this curve. The Henderson theory just doubles this energy consumption as it pictures double ionization throughout. This is denoted by the area under curve 2. Curve 3 gives the estimated energy loss calculated in this paper and lies somewhat between the two extreme values.

Thus one must conclude that this calculation of the ranges of α-particles by applying the concept of quantized energy exchanges to the classical dynamical treatment assuming free electrons initially at rest is even more inadequate to account for the phenomenon in He than has been supposed by previous writers. In how far the case of the other inert gases will show discrepancies of this order of magnitude when more accurately computed is not certain. From the way in which p for a given $Q(p)$ varies with the value of d where there are many electrons in a level one might conclude that a similar deviation as observed for He obtains.

One must then draw one of two conclusions. The energy of the α-particle is spent in processes other than ionization and radiation or else the energy is spent more freely in ionization than the theory here used permits. It was thought that some of this loss

might be accounted for through energy given the nuclei of the atoms on close encounters. Calculations indicate that the range as used in this paper is decreased by about 0.1 of 1 per cent. even if 180° deflections of the α-particle are allowed. These large deflections must be excluded, however, as the measurements of ranges of α-particles limit the deflections possible to minute angles. That an expenditure of energy in other ways than ionization and resonance does not occur to a great extent is also borne

FIG. 5.



out by recent measurements on the relative stopping powers of rare gases for α-particles by Gurney.[8] While the relative stopping powers of these gases compared with air may quite accurately be computed on the Henderson theory, the absolute observed values were on the basis of Fowler's computation for air about 1.8 the computed value. That is the ranges observed were about .55 of the computed value. Thus if other losses than ionization are the cause of this, the observed ionization in He should be .55 of the total possible ionization calculated from the available energy of the α-particle. Actually the amount of ionization observed for particles in He is about .8 the available energy. Hence the conclusion must be drawn that the second explanation

is the only possible one. This says that the energy losses must take place much more freely than the assumptions made in the theory used permit. As there is little doubt of the reality of the quantum conditions of energy transfer to electrons, the conclusion must be reached that it is in the application of dynamics that the error lies. It can probably be led back to the assumption that the electron was initially at rest and responded to the α-particle as if absolutely free. Since the velocity of the electrons in their orbits are of the order of those of the α-particles, it seems quite probable that the assumptions of electrons initially at rest and free are the cause of the error. The solution on the better hypothesis, however, leads to the complicated multiple body problems of dynamics which have not been solved.

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Perchloric Acid as an Analytical Reagent.—Prof. John H. Yoe, of the University of Virginia, reports several important applications of this substance. Analogy would suggest to most chemists that it would be easily decomposed, indeed, it would be regarded as dangerously explosive. It is, however, about the most permanent of the chlorine-oxygen series. Its aqueous solutions can be boiled without decomposition. Moreover, the dihydrate is a powerful dessicating agent. The sparing solubility of its potassium salt has long been known. As early as 1831, Serullas proposed it for determination of potassium, but his suggestion did not meet with immediate acceptance. When hot the acid is a powerful oxidizing agent on a great variety of organic substances such as gelatin, casein, blood. It is now being used in the laboratories of the University of Virginia for dehydration of silica, Kjeldahl procedure and determination of potassium. The details of these procedures are given by Professor Yoe in a communication to the *Ann. d. Chim. Anal. et Appl.* ([2], 1925, **7**, 193), from which this abstract is taken. The applications to silica dehydration and nitrogen determination were originally described by Willard and Cake, *JACS* (1920, **42**, 2208), and Mears and Hussey, *JIEC*

(13, 1054). The manufacture of perchloric acid by the electrolysis of hydrochloric acid was the subject of a patent issued in the United States (1,271,633) in 1918, to E. Walker.

The perchloric acid indicated in the procedures is of 60 per cent. strength, a grade that is now easily obtained in the market.

Determination of Potassium.—The presence of sodium salts does not interfere with this procedure. The principle of the method is extraction and not precipitation, being based on the insolubility of the potassium salt in strong alcohol while the sodium salt is soluble. Heavy metals must not be present, a condition that is usually attained without difficulty in ordinary analysis. The sample for analysis should be so treated as to have only sodium and potassium salts present. About half a gram is dissolved in the minimum amount of water, to a clear solution. Add 1 c.c. of perchloric acid and evaporate the mixture to a syrupy condition. Cool, add 15 c.c. of water and 2 c.c. of the acid and again evaporate. Add to the residue 15 c.c. of water and evaporate until heavy vapors of the acid cease to appear. Cool completely, add 20 c.c. of 97 per cent. alcohol containing 0.2 per cent. by weight of the acid, stir for five minutes in order to obtain a coarsely granular condition, allow to settle, decant through a tared gooch crucible, wash with a limited amount of the acid-alcohol, pour 20 c.c. of the same on the mass, stir, allow to settle, decant and remove the alcohol by evaporation. Dissolve the residue in 15 c.c. of warm water, add a few drops of perchloric acid, and heat until fumes of this appear, cool, add 1 c.c. of the alcohol-acid mixture, decant and evaporate a few drops to dryness. If a residue remains the process of extraction must continue until the decanted alcohol-acid no longer dissolves an appreciable amount of substance. Finally, treat the residue with 1 c.c. of the washing solution, transfer the salt to the gooch and wash with 2–3 c.c. of pure 97 per cent. alcohol, dry for forty-five minutes at 130° C. and weigh. The compound is, of course, potassium perchlorate. The error is only from 0.2 to 0.3 per cent. if the procedure is carefully carried out. It is not tedious or difficult, and avoids the cost of the platinum method.

Determination of Nitrogen.—Mears and Hussey, as noted above, first worked out the application of perchloric acid as an oxidant in the Kjeldahl method. Professor Yoe has employed the method for four years in the students' laboratory at the University of Virginia, for analysis of egg and blood albumins, dried blood and gelatin. The procedure is as follows: Transfer exactly a weighed portion, about 2 grams, of the sample to a 500 c.c. digestion-flask, and about 0.5 gram copper sulphate, 25 c.c. of sulphuric acid (1.84) add 2 c.c. of perchloric acid. Heat as usual under a hood, taking care that the flame does not strike above the liquid. Decoloration will usually occur in about five minutes, but the heating should be continued for fifteen minutes longer. The determination is then finished in the usual manner. Careful experiments by Yoe have shown that the process thus modified loses none of its accuracy. H. L.

A MATHEMATICAL THEORY OF THE DRYING OF WOOD.*

BY

FORDYCE TUTTLE.

DURING recent years extensive experimentation has been conducted at the U. S. Forest Products Laboratory, Madison, Wis., and elsewhere, in the artificial drying of wood. The object of much of this experimentation has been to adapt kiln temperature and humidity to the requirements of the various woods, so as to maintain at each successive stage of the drying process as fast a drying rate as possible without inducing the stresses which cause warping, checking, "casehardening," and other drying defects peculiar to wood.

Thus far the experimental work has been chiefly empirical, consisting in varying the drying conditions within the kiln and observing their effect on wood of a particular species and size. The formulation of drying schedules in this way for the innumerable kinds of wood stock that it is desirable to kiln-dry is a matter of exhaustive trial. If the physical law that governs the drying of wood could be ascertained, much of the empirical experimentation now necessary for the development of proper drying schedules would be eliminated.

In the following paper there is advanced a mathematical theory of the drying of wood, based by analogy on the Fourier heat-conduction analysis. Experimental data in support of the theory are also presented. The idea of following up the analogy was suggested by W. Karl Loughborough, of the Forest Products Laboratory. The data were obtained in coöperation with Mr. Loughborough, using an experimental kiln which he had built at the Forest Products Laboratory for the study of isolated factors in the drying of wood, and which provided particularly accurate control of temperature and humidity.

HYPOTHESIS.

It is known that when different parts of a piece of wood are at different moisture contents, moisture transfuses from the wetter to the drier regions. The fundamental hypothesis here

* Communicated by the Department of Physics, University of Wisconsin.

assumed is that the rate at which this transfusion takes place, transversely with respect to the wood fibres or elements, is proportional to the slope of the moisture gradient in that direction.

This assumption allows us to write for the flow of moisture transverse to the grain, a differential equation by analogy with the Fourier heat-conduction equation for unidirectional flow, *viz.*,

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2}, \qquad (1)$$

where θ is moisture content expressed as a percentage of the oven-dry weight of the wood, and h^2 is what may be called the "transfusivity" (across the grain) of the species in question.

The assumption that the rate of transfusion of moisture is proportional to the slope of the moisture gradient must, of course, be justified by experimental evidence before it is adopted as valid. If it is shown to be valid, it becomes necessary to evaluate h^2 in order to make the theory useful. This paper outlines a method for testing the theory and at the same time for evaluating h^2. Actual drying data for Sitka spruce heartwood are presented, from which a satisfactory value of h^2 is derived.

COMPUTATION OF h^2—FIRST APPROXIMATION.

Taking equation (1) as the drying equation of a large piece of wood drying from one (plane) face and assuming as boundary conditions

$$\theta = 0^1 \text{ at } x = 0$$
$$\theta = \theta_0 \text{ when } t = 0$$

we can solve as in the Fourier case of the semi-infinite solid and obtain the equation,[2]

$$\theta = \frac{2\theta_0}{\sqrt{\pi}} \int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2} d\beta \qquad (2)$$

In the case of a very thick plank initially of uniform moisture

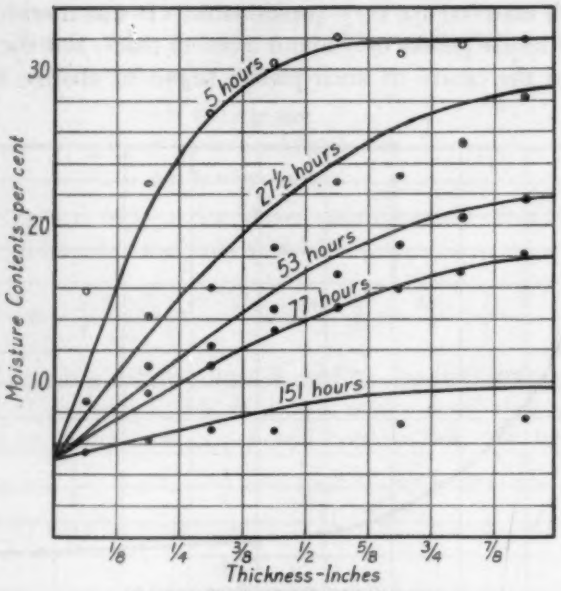
[1] The moisture content scale can be shifted so that any equilibrium moisture content may be called the zero of moisture content. If a be the equilibrium moisture content under a given set of conditions the θ_0 in the second boundary condition ($\theta = \theta_0$ when $t = 0$) would be the actual moisture content in per cent. minus a.

[2] For the theory involved and the steps necessary in the derivation of this equation, see "Mathematical Theory of Heat Conduction," Ingersoll and Zobel. (Ginn and Co., 1913.)

content θ_0, drying from one face, we should expect equation (2) to express approximately the moisture content at any time t at any distance x from the drying surface.

Under ideal experimental conditions, h^2 could be evaluated

FIG. 1.



Moisture distribution in 2-inch spruce.
Initial moisture content, 32%; equilibrium, 5%; drying conditions, 160° F., 30% R. H.; curves computed for $h^2 = 0.0053$.

from moisture gradient data with the aid of equation (2). Thus for a plane in the wood to reach a moisture content $\frac{1}{2}\theta_0$, the "probability integral," $\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2h\sqrt{t}}} e^{-\beta^2} d\beta$ would have to be equal to 0.5, which makes the upper limit, $\frac{x}{2h\sqrt{t}}$, equal to about 0.48;[3] h^2 could then be evaluated from the equality

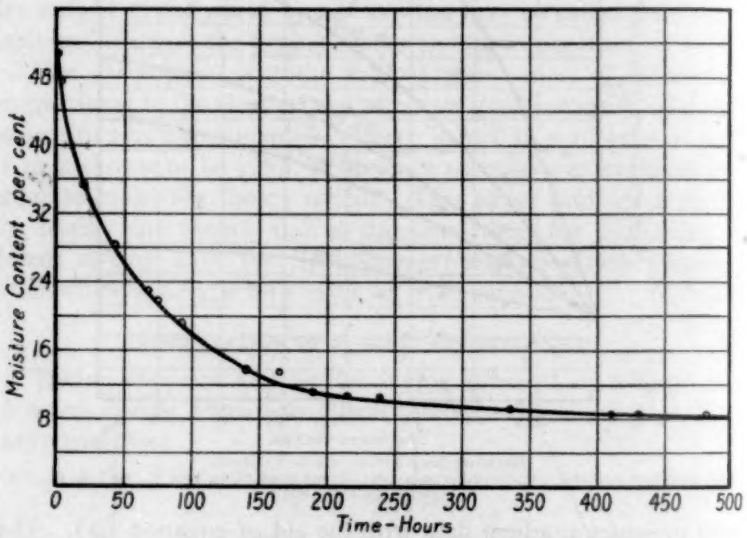
$$h = \frac{x_1}{2\,(0.48)\,\sqrt{t_1}}$$

where x_1 is the distance of the plane which has reached a moisture content $\frac{1}{2}\theta_0$ in time t_1.

[3] See Table of Values of Probability Integral.

Experimental Difficulties.—A plank of finite thickness drying from both faces can be considered as drying like a semi-infinite solid only so long as the moisture content at the centre of the plank remains very nearly constant. Thick planks were found to check (split open at the surface) rather badly, and drying data derived from such material are very questionable. It was therefore found necessary to use planks only about 2 inches thick, and the moisture content in the centre of such planks began to change somewhat

FIG. 2.



500-hour drying curve for 2-inch spruce.
Initial moisture content, 51%; equilibrium, 8%; drying conditions, 160° F., 75% R. H.; curve computed for $h^2 = 0.0053$.

within a few hours. But values of h^2 computed by the method outlined above from gradient data collected in the first few hours of drying, can be considered only rough approximations, since the experimental error involved may be large. From such approximations, however, it is possible to determine the *order of magnitude* of h^2.

From actual moisture distribution data on 2-inch Sitka spruce heartwood after five hours' drying at 160° F. and 30 per cent. relative humidity, h^2 was computed to be 0.0064, where distance is in inches, time in hours, and moisture content in percentage of dry weight of wood.

Once the order of magnitude of h^2 has been established, it is possible to find more accurate values of h^2 from long-time gradients by the method outlined below.

FINAL COMPUTATION OF h^2.

In the final method of evaluating h^2, the differential equation (1) was solved under boundary conditions more nearly representing those for an actual plank, *viz.*,

$$\theta = 0 \text{ at } x = 0$$
$$\theta = 0 \text{ at } x = l \text{ (thickness)}$$
$$\theta = \theta_0 \text{ when } t = 0$$

The solution of (1) under these conditions is similar to the Fourier equation for the slab, which is

$$\theta = \frac{2}{l} \sum_{m=1}^{m=\infty} \left\{ e^{-\frac{h^2 m^2 \pi^2 t}{l^2}} \sin \frac{m\pi x}{l} \int_0^l \sin \frac{m\pi \lambda}{l} \, d\lambda \right\} \theta_0. \tag{3}$$

For a 2-inch plank, we have found by the first approximation that h^2 is of the magnitude of 0.005, and a good approximation of the above expression for θ will contain but one term when $t > 50$ and $x > \frac{1}{2}$. For $x = \frac{3}{4}$ inch and $t = 77$ hours, for example, formula (3) may be rewritten

$$\theta = e^{-\frac{h^2 \pi^2 77}{4}} \left(\sin \frac{3\pi}{8} \right) \frac{4}{\pi} \theta_0 \tag{4}$$

where θ_0 = original moisture content of wood; whence, if θ be known from measurement, the value of h^2 is easily determined.

AGREEMENT OF THEORY WITH EXPERIMENTAL DATA.

A careful experimental kiln run on 2-inch Sitka spruce planks with moisture-resistant coatings on edges and ends yielded the data shown in Fig. 1. The wood had an initial moisture content of 32 per cent., and after seventy-seven hours of drying at 160° F. and 30 per cent. relative humidity had a moisture content at a depth ¾ inch from the surface of 16.5 per cent. The moisture content of wood in equilibrium with the given temperature and humidity conditions is known to be 5 per cent. Hence, from equation (4) and the observations just stated, the final value of h^2 for Sitka spruce for this drying condition was computed as follows:

$$\theta = (16.5 - 5) = 11.5$$

$$e^{-q}\left(\sin\frac{3\pi}{8}\right)\frac{4}{\pi}(32 - 5) = 11.5$$

$$q = 1.00$$

$$q/t = \frac{h^2\pi^2}{4} = \frac{1.00}{77} = 0.013$$

$$h^2 = 0.0053$$

This value of 0.0053 for h^2 was substituted in formula (3), which was then used in the computation of the theoretical moisture distribution from the centre to the surface of the 2-inch plank after drying periods of 5, 27½, 53, 77, and 151 hours. The resulting curves are plotted, along with the moisture contents as determined by actual tests on successive ⅛-inch laminæ, in Fig. 1.

In Fig. 2 is the predicted drying curve (average moisture content against time), checked against experimental data, for a 2-inch spruce specimen having an initial moisture content of 51 per cent. and drying (from two faces only) for 500 hours at 160° F. and 75 per cent. relative humidity. The formula used in computing the curve was obtained by the integration of equation (3) with respect to x between limits of $x = 0$ and $x = l$, and division by the thickness l. Thus,

$$\theta_{Ave} = \frac{2}{l^2}\sum_{m=1}^{m=\infty}\left\{e^{-\frac{h^2m^2\pi^2t}{l^2}}\int_0^l \sin\frac{m\pi x}{l}\,dx\int_0^l \sin\frac{m\pi\lambda}{l}\,d\lambda\right\}\theta_0$$

whence,

$$\theta_{Ave} = \sum_{m=1}^{m=\infty} e^{-\frac{h^2m^2\pi^2t}{l^2}}\left[\frac{4}{m^2\pi^2}\right]\theta_0. \qquad (5)$$

Marking Ink for Porcelain.—The Coors Porcelain Company of Golden, Colo. (*Ind. and Eng. Chem. News Edition*, 1925, III, No. 18, 3), furnishes the following formula for a marking ink for chemical porcelain: 18.8 grams of commercial black oxide of cobalt, 1.2 grams of bismuth subnitrate, 15 c.c. of oil of turpentine, and 15 drops of Dresden thick oil. The ingredients are thoroughly mixed by grinding. J. S. H.

Dyes as Microchemical Reagents.—H. J. CONN, of the New York Agricultural Experiment Station at Geneva (*Jour. Chem. Education*, 1925, **2**, 753–755), points out that the use of stains in histology and other branches of biology is really based on chemical affinities between the dye in the stain and the particular portion of the animal or vegetable cell which combines with the dye. In other words, all forms of staining prior to microscopic examination are microchemical reactions. J. S. H.

SOFT X-RAYS FROM CERTAIN METALS.*

BY

C. T. CHU.

Research Section, Randal Morgan Laboratory of Physics, University of Pennsylvania.

MANY theories of atomic structure, such as those of Thomson, of Rutherford, of Langmuir, and of Lewis, have been advanced in recent years to explain phenomena in certain fields of chemistry and physics. The theory by Bohr has found the widest application, especially in spectroscopy. According to Bohr, an atom consists of a nucleus of net charge eZ with Z electrons, each carrying a charge of $-e$, revolving around it in different groups of orbits designated as the K, L, M, N, O and P groups counting from the innermost. There is only one member in the K group, but three in the L group, five in the M group, seven in the N group, five in the O group and three in the P group. The number of groups occupied by the electrons depends on Z, the number of electrons of the normal atom. Z is known as the atomic number, characteristic of the atom and hence of the element. The hydrogen atom, having an atomic number 1, is the simplest atom with only one electron in the K group. The complexity of the atom increases with heavier elements up to uranium, the heaviest known, which has an atomic number 92. In the copper atom (At. No. 29), for example, the K, L and M groups are fully occupied, respectively, by two, eight and eighteen electrons, leaving the twenty-ninth electron in the N group.

Based on such an atomic model, optical spectral lines are explained as radiations produced by rearrangements of the exterior electrons after displacements, while the X-ray spectral lines are developed by transitions of electrons of the inner orbits. A K line, for instance, is emitted when an electron from an outer orbit falls back into the K orbit, in which a vacancy has previously been produced by an energy absorption. The quantum theory, when applied to this case, requires the condition that the difference of energy between two successive groups must be a quantum. The K group corresponds to the one-quantum energy level, the L group corresponds to the two-quantum energy level, etc.

* Communicated by Prof. C. B. Bazzoni.

The knowledge of the close relationship between X-radiation and atomic structure is not at all new. Barkla, soon after the discovery of X-rays, found that most substances emit X-rays characteristic of themselves and a relation between the "hardness" of the X-rays emitted by an element and its atomic weight was established. Since then, better devices have been made for the study of X-rays. With Laue's discovery, the wave-lengths of X-ray spectrum lines could be determined by the use of crystals, which behave towards the radiation more or less like a three-dimensional optical grating. The extent to which this method is applicable depends on the lattice constant of the crystal, d, since

$$\lambda = \frac{2d}{n} \sin\theta$$

$$\lambda \leq \frac{2d}{n} \leq 2d$$

As $d = 2.814 \times 10^{-8}$ cm. for rock salt, wave-lengths longer than 5.628 Å. will not be measurable with a rock salt crystal. Measurements with the crystal grating spectrometer extend from $\lambda = 12$ Å. to $\lambda = 0.1$ Å. To measure longer wave-lengths a new crystal with a longer grating distance must be discovered.

Optical and ultra-violet wave-lengths are measured by the ordinary spectrometers with gratings, Lyman having thus penetrated to 540 Å. and Millikan with a special line grating to 144 Å. In spite of these recent achievements there is, in fact, a considerable gap between the longest X-ray which can be measured with crystals and the shortest ultra-violet ray which can readily be handled with line gratings. This gap constitutes the soft X-ray region. The wave-lengths lying in that region are of particular interest and importance in the present state of theory on atomic structure, since from them we can learn the arrangement of electrons in the subperipheral orbits where the dynamical laws are presumably those appropriate neither to the inner orbits nor to the most exterior orbit. Our present ignorance of conditions in these orbits is due to the inadequate experimental determination of the corresponding emission lines and absorption edges. The present investigation has to do with critical potentials for the excitation of radiations which are ascribed to electron shifts amongst these particular orbits.

METHOD OF PROCEDURE.

As has been stated, the soft X-ray region cannot be completely explored from the ultra-violet end, due to the lack of adequate line gratings. Nor can it be studied by the ordinary X-ray spectrometers. The experimental difficulties are increased because of the low penetration of the rays, which are readily absorbed by all materials. Holweck found that films of celluloid even as thin as 2×10^{-5} cm. were over 97 per cent. opaque at 300 Å. Using such a film as a window for an ionization chamber he was, however, able to detect radiation developed by an electron stream under 70 volts potential, corresponding to $\lambda = 175$ Å.

To carry out general investigation in the soft X-ray region, all windows in the path of the rays must be eliminated, and since this makes the use of ionization chamber methods impossible, it has been the general practice to detect the radiation by the observation of the photo-electric emission stimulated by it from a metallic disc as originally employed by Dember.[1] Windows having been eliminated, a device must be introduced to prevent the drifting of electrons and stray positive ions from the X-ray chamber to the detecting chamber.

The various workers in this field have used such "ion traps" of a variety of designs, but all have been made on the same principle. Two, or two sets of, metallic conductors are placed in the path of the radiation and charged electrically, thus creating an electrostatic field. Such an arrangement has been found efficient when properly designed and charged. Dember [2] used a system of gauzes parallel to the photo-electric disc. Richardson and Bazzoni [3] used a metallic wire with a coaxial cylinder, creating a radial field when charged. They [4] later used two parallel plates close together, similar to J. J. Thomson's [5] arrangement. Kurth [6] used seven diverging plates charged alternately positively and negatively. The writer used an axial wire in a cylindrical tube which had another tube of greater diameter attached to one end.

[1] H. DEMBER, *Verh. Deutsch. Phys. Ges.*, **13**, p. 601, 1911.
[2] See 1, also H. DEMBER, *Verh. Deutsch. Phys. Ges.*, **15**, p. 560, 1913.
[3] RICHARDSON and BAZZONI, *Phil. Mag.*, **32**, p. 426, 1916.
[4] RICHARDSON and BAZZONI, *Phil. Mag.*, **34**, p. 285, 1917.
[5] J. J. THOMSON, *Phil. Mag.*, **28**, p. 620, 1914.
[6] KURTH, *Phys. Rev.*, **18**, p. 461, 1921.

The cylindrical plate thus functioned also as an electrostatic shield for the discharge.

The mode of investigation, in its elements, is to bombard a target of the material being studied with electrons in the highest attainable vacuum and to examine the photo-electric emission from a second electrode exposed to the radiation. The setting in of photo-electric emission from the second target is evidence of the stimulation of an X-radiation from the bombarded target. The energy of the bombarding electrons being known, we have a measure of the energy necessary to produce a radiation from the investigated element.

By this method, a number of elements (beryllium, boron, carbon, sodium, magnesium, aluminum, silicon, phosphorus, sulphur, potassium, titanium, iron, nickel, copper, tungsten, and molybdenum) were studied by Richardson and Bazzoni,[7] Kurth,[8] Hughes,[9] Mohler and Foote,[10] and Holtzmark.[11] During the course of the present investigation, results of similar work have been made known by McLennan and Clark,[12] Rogers,[13] Rollefson,[14] Boyce,[15] Thomas,[16] and others.

APPARATUS.

In the present investigation, a quartz tube (Fig. 1) made by the Cooper Hewitt Electric Company, Hoboken, was used. Referring to Fig. 1, the source of thermions is a tungsten filament, *F*, 0.003 mm. in diameter. The element under investigation is supported at *T*, directly in front of the photo-electric target *E*, with the ion trap *C* and *C'* between. Chamber *D* is the detecting chamber.

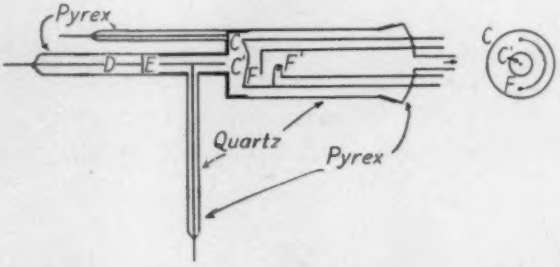
In the cleaning-up process this device showed marked advantages. The cylindrical tube forming one element of the condenser could readily be heated up by an induction furnace. The target under investigation was bombarded by thermions from the fila-

[7] RICHARDSON and BAZZONI, *Phil. Mag.*, **42**, p. 1015, 1921.
[8] See 6.
[9] HUGHES, *Phil. Mag.*, **43**, p. 145, 1922.
[10] Bureau of Standard Sci. Paper No. 425.
[11] HOLTZMARK, *Phys. Zeit.*, **23**, p. 252, 1922; **24**, p. 225, 1923.
[12] MCLENNAN and CLARK, *Roy. Soc., Pro.*, **102**, p. 389, 1923.
[13] ROGERS, *Camb. Phil. Soc., Pro.*, **21**, p. 430, 1923.
[14] ROLLEFSON, *Science*, **57**, p. 562, 1923; *Phys. Rev.*, **23**, p. 35, 1924.
[15] BOYCE, *Phys. Rev.*, **23**, p. 575, 1924.
[16] THOMAS, *Phys. Rev.*, **25**, p. 322, 1925.

ment F' and not from F, which if used, as with the old tube,[17] would sputter tungsten from the hot filament on the surface of the target facing the photo-electric disc. This target would then undoubtedly emit X-rays characteristic of tungsten as well as of the element itself. Unless the element under investigation was tungsten, the results obtained would not be conclusive. Besides, a filament used for bombarding purposes is usually overheated and its life shortened considerably. The introduction of the extra filament F' permits a longer usage of F in the investigation proper.

The apparatus was connected through three liquid air traps, one of which contained charcoal, to a large all-metal diffusion

FIG. 1.



pump backed by two oil pumps maintaining a vacuum of 10^{-6} mm. of mercury or better. In the cleaning-up process, the central part of the tube was heated to redness by blast lamps. The shielded metal parts, not reached by the induction furnace, were heated by electron bombardment. For the latter process, a two-thousand volt d.c. generator was available.

Except in the very early part of the experiment, all the electrodes in the apparatus were tungsten rods sealed with pyrex, then fused to the quartz by Cooper Hewitt joints. The stoppers were cemented without grease with Bank of England sealing wax.

The filament (F, Fig. 1) was heated with a small current of 0.4 or 0.5 ampere, the voltage drop being 1.8 or 2.0 volts. The driving potential, filament to target, was obtained from a potential divider placed across a high-capacity storage battery. The potential used was usually below 120 volts. Working with low voltages and with a filament of low temperature, the thermionic current was less than eight microamperes. The use of small thermionic

[17] BAZZONI and CHU, JOUR. FRANK. INST., 197, p. 183, 1924.

current prevented tungsten sputtering on the target and effects due to space charge.

The photo-electric disc in the tube was made of nickel. It was carefully polished before use, and cleaned by heat after introduction into the vacuum. This disc was connected to a Dolezalek electrometer with a sensitivity of about 800 mm. a volt. The

FIG. 2.

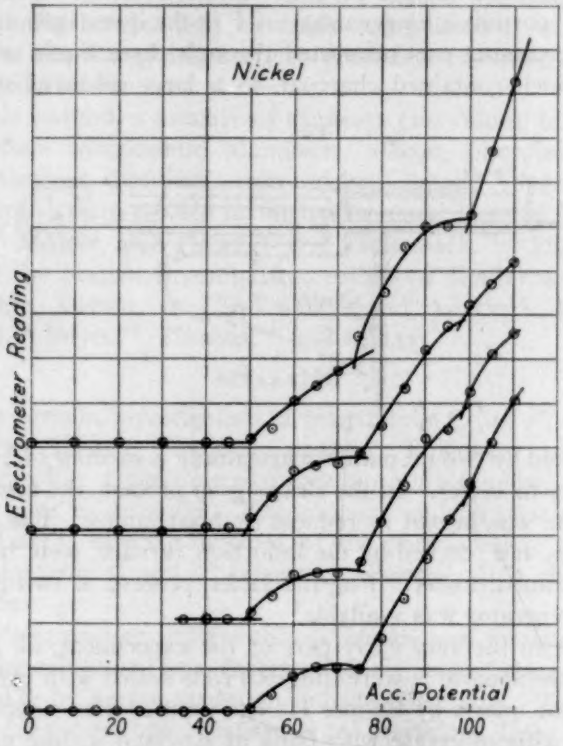
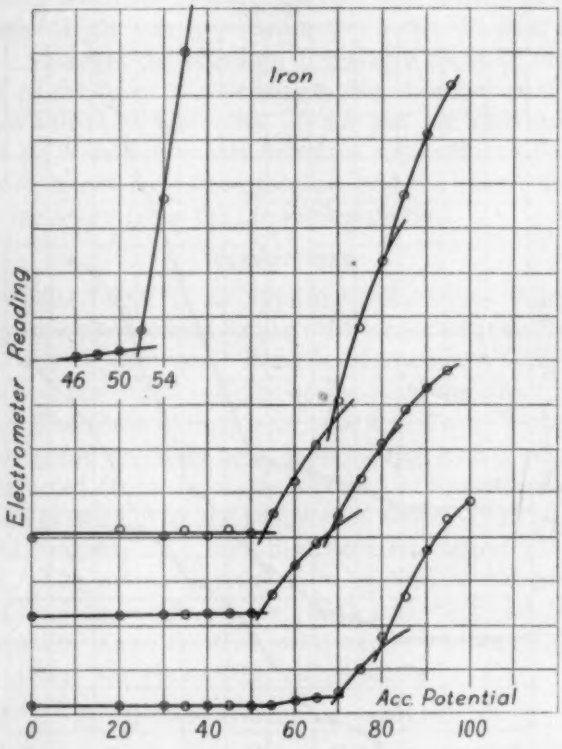


photo-electric current was always so small that a high-resistance shunt was not found necessary. It was thought to be a better arrangement to expose the photo-electric disc to a small portion of the total radiation and to use a sensitive electrometer than to get almost all the radiation and to measure the photo-electric emission resulting from it with a shunted electrometer. The ion trap was designed with this idea in mind. The photo-electric disc, 7 mm. in diameter, was set 3½ cm. away from the anode, subtending a

small solid angle. The breaks found in the curves were, under these conditions, few in number, prominent and more definite than those obtained by workers using large solid angle methods. The great difficulty in this kind of work is to protect the photo-electric disc from stray effects due to ion leakage or to shifting electro-

FIG. 3.



static charges. These difficulties multiply with the solid angle of exposure.

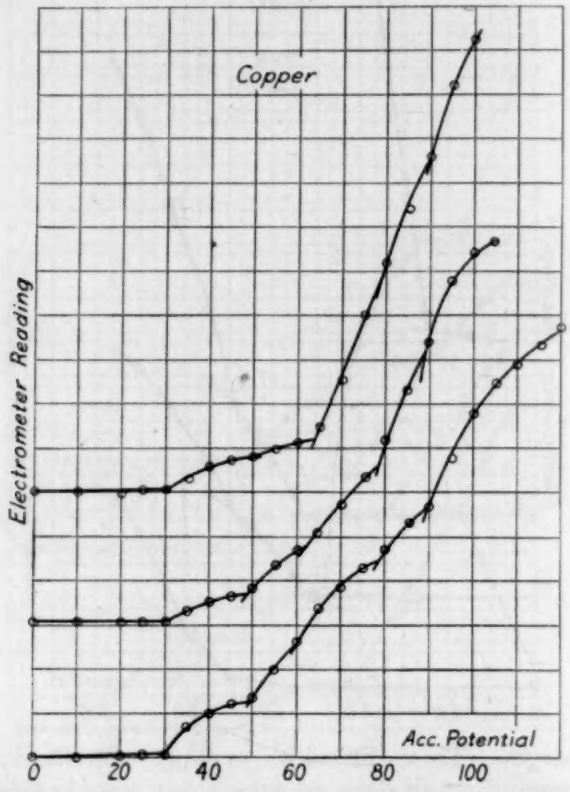
RESULTS.

The observations made in this work were of the photo-electric current corresponding to a certain accelerating potential from filament to the target. The former, represented by the electrometer deflection, was plotted against the latter, and the points of change of slopes of the curves were inferred to be points at which X-radiations set in. Some investigators have, with advantage, plotted

electrometer deflections per unit thermionic current against the accelerating potentials. In the present work, however, low-filament temperature was used and saturation thermionic current was maintained, so that either method of plotting would give the same shape to the curves.

Nickel.—Nickel was the first element investigated. The

FIG. 4.



curves obtained are shown in Fig. 2. The breaks occurred at 49, 75 and 98 volts (uncorrected values read directly from the experimental curves).

Iron.—Iron was the next element to be studied. Breaks at 52, 68 and 80 volts are easily recognized in Fig. 3. The values are uncorrected.

Copper.—For copper, radiations set in at 30, 49, 62, 79, and 90 volts, uncorrected. Sample curves are shown in Fig. 4.

Cobalt.—To complete the series of elements of atomic numbers 26, 27, 28 and 29, cobalt is yet to be studied.

Selenium.—Selenium is an attractive substance to investigate. Due to its low melting-point, it may be possible to study it in both the solid and vapor states. However, the attempt to investigate it in the solid state has not been successful, with the present arrangement. Even when the filament current was greatly reduced and the selenium target was two centimetres away, the heat was great enough to vaporize the selenium forming a metallic film around the inside of the tube. No deflection was observed even when the target was raised to 120 volts. This was not conclusive, however, that no X-radiation sets in below 120 volts, as the filament temperature might have been too low and the target might have been too far away from the photo-electric disc.

CORRECTIONS.

Corrections ought to be applied to the above values due to voltage drop in the filament, contact difference of potential and the initial velocity of electrons. Because of low-filament temperature the correction for initial velocity of the electrons was found to be negligible. The filament drop was 1.8 volts. The end correction to cover this would probably be a tenth of that value, say 0.2 volt, to be subtracted from the anode voltage. The contact difference of potential between iron and tungsten is about -0.4 volt; between nickel and tungsten -0.6 volt, and between copper and tungsten -0.1 volt. The net corrections to be applied would probably be -0.6 volt for iron, -0.8 volt for nickel, and -0.3 volt for copper. The uncertainty as to the true value of the contact difference of potential makes the corrections questionable.

INTERPRETATION OF RESULTS.

The interpretation of results of the character of those above cited is not entirely obvious. A sufficient divergence has developed amongst different investigators in this field, both in results and in modes of correcting results, to indicate the need for careful consideration. In the first place, it is recognized that break points on the "photo-electric emission-voltage" curves indicate merely radiation excitation potentials of the bombarded target or of the radiator. It is by no means necessary to suppose that the frequency of the radiation emitted from the target will be given by a simple application of the $eV = h\nu$ relation where eV represents

the energy of the impinging electron. That this relation is valid in the hard X-ray region to give the stimulation potential of an X-ray series, and that it is also valid in the optical region to give stimulation potentials of successive series lines up to and including the ionization limit is well known, but the conditions in the orbits involved in soft X-ray development cannot be taken necessarily to afford an analogous set of phenomena. Characteristic soft X-rays are developed by electron shifts from orbits which lie between the valence orbits and the deep-lying, fully occupied, hard X-ray orbits. Neither the dynamical conditions in these intermediate orbits nor the general configuration of the electrons in them is known as definitely as for the other groups.

The most direct interpretation of results such as are here obtained is to assume the existence of intermediate orbits with energies associated with the *M*, *N*, *O* and *P* orbits by workers using crystal grating methods, and to endeavor to allot each stimulation potential to an ionization process in which an electron is extracted from the corresponding orbit. We may as a result of this ionization process expect a polychromatic radiation consisting of the entire series for the orbit affected. This polychromatic radiation would be responsible for the observed increase in photo-electric current. The number of breaks in the current curve with increasing impact voltage would then be limited to the number of different orbits reached by the voltage range. This number in all cases would be small. If the number of breaks could be shown to be consistently identical with the number of orbits thought on the basis of the other evidences to exist, we should have then valuable evidence on the energy content of these orbits which can be only roughly approximated to by the extrapolation methods used by Coster, Hjalmar, Walter and others.

In the next place, we may complicate these possibilities by assuming, with Kossel, a number of virtual peripheral orbits to which an electron may be thrown. Such an assumption seems necessary in accounting for the known fine structure of the *K* and *L* absorption edges. If such peripheral orbits exist for soft X-ray stimulation, there should be several breaks in the photo-electric emission curve for each orbit penetrated, an outburst of radiation taking place whenever the voltage applied exceeds the necessary value to fill an extra orbit. This would account for the existence

of a considerable number of stimulation potentials lying close together, as has been reported by some workers.

Or, again, impacts of electrons may cause a complicated readjustment amongst the loosely bound electrons in the intermediate orbits, the energy values of which are known to differ very little from each other. Some of these orbits are incompletely filled in the normal atom. The stimulated radiation may then be the integrated result of an atomic readjustment, during which some of the vacant places are temporarily occupied.

We note, therefore, as equally probable possibilities, that the number of breaks on bombarding potential-photo-electric emission curves may be limited (*a*) to a small number corresponding to the number of orbits, as is in the case with the excitation levels for hard X-ray; (*b*) to a greater number of possible transfers into outer virtual orbits (peripheral orbits) similar to the resonance and excitation lines observed in connection with *L* levels; (*c*) to an indefinite number of possible transfers into the incomplete outer shells.

To solidify speculation in this field, it is obviously desirable to measure the wave-lengths of the radiations produced by these low-voltage impacts.

Continuing the analysis of the experiment farther, it is safe to assume that when the electrons fall on the anode, there will be produced a general radiation with a maximum value given by the $eV = h\nu$ relation. When a voltage value, capable of lifting an electron to an outer orbit or out of the atom, is reached, the displaced electron on falling back will give out radiation of a frequency determined by the energy difference of the two states. At these critical potentials, extra radiation will therefore be produced on the continuous background. At any given voltage, the intensity of the radiation will be proportional to the area of a curve representing this double effect. The radiation intensity will therefore increase with the voltage, but at a variable rate.

The magnitude of the photo-electric current from the detecting disc will depend on the amount of radiation falling on the disc in unit time and, consequently, there will be increases in the photo-electric current when the potentials characteristic of the anode are passed. However, this current should show also the characteristics of the material of the photo-electric target. The electrons

constituting the emission are, according to Becker,[18] emitted in groups; the large majority having a velocity determined by the $h\nu$ values of the electrons in the original discharge, and the others having velocities corresponding to this $h\nu$ value minus the work of extracting electrons from the successive levels in the atoms of the photo-electric disc. When, therefore, the $h\nu$ value is exactly sufficient to cause one of these orbit changes, the photo-electric emission will presumably change. We ought, consequently, to find on the photo-electric emission curve, breaks in slope characteristic of both target and anode levels.

The concordance of observed values, as determined by different investigators, amongst different observers and against the values to be expected from X-ray theory (provided the critical

FIG. 5.

Element.	Iron.	Nickel.	Copper.
At. no.	26	28	29
Orbits	M_{III} N_{III}	M_{III} N_{III}	M_{III} N_{II}
ν (Walter)	5.85 0.46	5.13 0.37	3.89 0.25
$\overline{R}$ (Bohr and Coster)	5.2	3.3	4.0
Corresponding volts : W.	80.7 6.0	71.0	54.0
volts : B. & C.	72.0	45.0	55.0
Observed volts:			
Chu	52, 68, 80	49, 75, 98	30, 49, 62, 79, 90
Kurth	50		106
Horton	73	104	112
Foote & Mohler		82	
Rollefson	46.8 81.7 95.4		
Thomas	7.3 11.1 14.1 16.5 19.4 24.3 28.8 34.3 41.2 48 51.3 54.6 62 75.3 82.7 94.8 etc.		

potentials are ionization values for the levels specified) is shown in the tabulation (Fig. 5). The concordance amongst the values is seen to be so slight that it is without critical significance. Referring to the results obtained in the present investigation, the photo-electric target for the three metals covered was in all cases nickel, which is shown to have a critical point at 49 volts. The breaks 52 for the iron anticathode and 49 for the copper anticathode may

[18] BECKER, *Phys. Rev.*, **24**, 478, 1924.

possibly therefore be characteristic of the photo-electric target. Leaving this feature out of consideration the results, considered conservatively, establish that for iron (against a nickel detector) radiation begins close above 50 volts, and increases sharply just below 70 volts; for nickel (against a nickel detector) radiation begins close below 50 volts and increases sharply at 75 volts; for copper (against nickel) radiation begins at 30 volts and increases sharply at 80 volts and 90 volts. These results are not entirely inconsistent with what might be expected from the position and order of the particular elements in the periodic table. The necessity for further work in this field is, however, very apparent.

In conclusion, the writer wishes to express his indebtedness to Prof. C. B. Bazzoni, at whose suggestion the work was undertaken, for his help and encouragement during the investigation, and for his criticism of the results.

The Ether-drift Experiments of Dayton C. Miller.—An account of these was given on page 259 of this JOURNAL for August, 1925. Observations made on Mount Wilson indicate that, at that altitude, the earth and the adjacent ether do not move together but that, on the contrary, they show a relative velocity of about one-third of the earth's orbital velocity, 9 or 10 km. per sec. At the lower level of Cleveland, Ohio, the relative velocity is only one-third as much as on the mountain 1.73 km. high. The earth loses its grip on the ether with increasing altitude.

The scientific consequences of these experimental data are far-reaching. At the assembly of the International Physics Union in July of this year, H. A. Lorentz is quoted as holding that "the preliminary results were in complete contradiction to the relativity theory as proposed by Einstein, and if finally established they would mean the end of relativity in that form." (*Science*, Aug. 7, 1925.) Einstein himself says: "If Doctor Miller's results should be confirmed, then the special relativity theory, and with it the general theory in its present form, falls. Only the equivalence of inertia and weight remain, which would lead to an essentially different theory." According to the astronomer, A. S. Eddington, a difference in the value of ether-drift with altitude would cause the apparent position of a star to vary with the altitude of the place from which it is viewed. Since no such difference has ever been noticed an argument is adduced against the correctness of Miller's results. Dr. Ludwig Silberstein, however, thinks that a special theory of the ether would permit the reconciling of Miller's observations with the lack of change in star positions with elevation. With this opinion Professor Eddington is by no means in agreement saying: "Under

these circumstances nothing remains but to await more complete publication of Miller's results. Then it is to be hoped that a correct decision will develop." (*Science*, July 31, 1925, and *Nature*, July 25, 1925.)

It should be remembered that Professor Miller is no novice in this type of work. A score of years ago he collaborated with Edward W. Morley in repeating in Cleveland the famous Michelson-Morley experiments on ether-drift. Besides this he has made valuable contributions in acoustics and in harmonic analysis. All these things guarantee the accuracy of his observations. "The final test of these observations," says Professor Miller, "is whether they lead to a rational and wholly consistent indication of a constant motion of the solar system in space, combined with the orbital motion of the earth and the daily rotation on its axis." (*Nature*, July 11, 1925.)

G. F. S.

The Making Visible of Rapid Longitudinal Vibrations Produced by Piezo-electric Means in Rods of Crystals. E. GIEBE and A. SCHEIBE. (*Zeit. f. Physik*, **33**, Aug. 8, 1925.)—The piezo-electric effect in crystals, discovered by J. and P. Curie in 1885, has two aspects. When an appropriate crystal is elastically deformed one end of it becomes positively and the other negatively charged. Conversely, when the crystal has imposed upon it a state of electrical polarization, it suffers elastic deformation. By the application of these relations it has been possible to produce a piezo-electric telephone. In 1922 W. G. Cady showed that a rod of crystal could be made to vibrate with high frequency by the application of an alternating electric field of such frequency as to correspond to one of the modes of vibration of the crystal. In this paper the state of vibration of the crystal is shown to produce a visible, luminous effect. The crystal is located in a vacuum of 10 or 15 mm. of mercury and the alternating field is brought to bear apparently perpendicular to the length of the rod, whereupon a luminous discharge appears along the crystal. Its intensity is greatest at the middle and least at the ends. In a particular case the frequency of the vibrations was 27,400 per second and the wave-length of the exciting electric wave was 10,950 metres. Owing to the damping of the rod the state of resonance declared itself with great sharpness. The exciting frequency for which the luminous effect stretched for the greatest distance along the rod could be determined with an accuracy equal to 1/10,000 of its value. By less simple arrangements of the electrodes for the application of the alternating field the rod was made to vibrate in modes of overtones and correspondingly more complicated luminous effects were achieved. Even with air at normal pressure, provided that the distance from the rod to the electrodes was very small, the points of maximum condensation and rarefaction could be made to declare themselves by an irregular emission of electric sparks.

G. F. S.

SOLAR RADIATION DURING THE TOTAL ECLIPSE OF JANUARY 24, 1925.*

BY

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Physical Laboratory, Cornell University.

A FEW days prior to the recent total eclipse of the sun it was suggested by Prof. C. C. Bidwell that it ought to be a comparatively simple matter to adapt a Fery radiation pyrometer to observations of the relative values of the total radiation. A pyrometer of this type, as manufactured by the Taylor Instrument Company, was used in connection with a Leeds and Northrup moving coil galvanometer of 385 megohms sensitivity, 575 ohms resistance and 2.2 seconds period. The scale distance used during observations was .5 metre, so that the sensitivity actually employed was 192.5 megohms for one millimetre scale deflection. The Fery instrument consists of a concave mirror with a cell at its focus containing a thermocouple. The opening may be wide open or partly closed with an adjustable sector-shaped slit to vary the transmission. If the instrument is used for measuring temperature, it is essential that the image of the hot object be larger than the cell, but if variation in radiation from a given object (*e.g.*, the sun) be desired, without regard to temperature, it is equally important that the image be smaller than the cell. This requirement was fulfilled in the case at hand and the image of the sun was enough smaller than the cell to make it easy to keep the instrument "on the sun" during observations. The galvanometer was connected to the pyrometer through a reversing switch so that the zero could be checked occasionally. The zero remained constant throughout the work.

The equipment was set up on the third floor of Rockefeller Hall near an open window. One observer sighted the pyrometer on the sun and recorded galvanometer deflections as they were read by a second observer. Time was taken from a watch which was but a few seconds off. The author was assisted in the observational work by Dr. M. Scott.

* Communicated by Prof. C. C. Bidwell, Department of Physics.

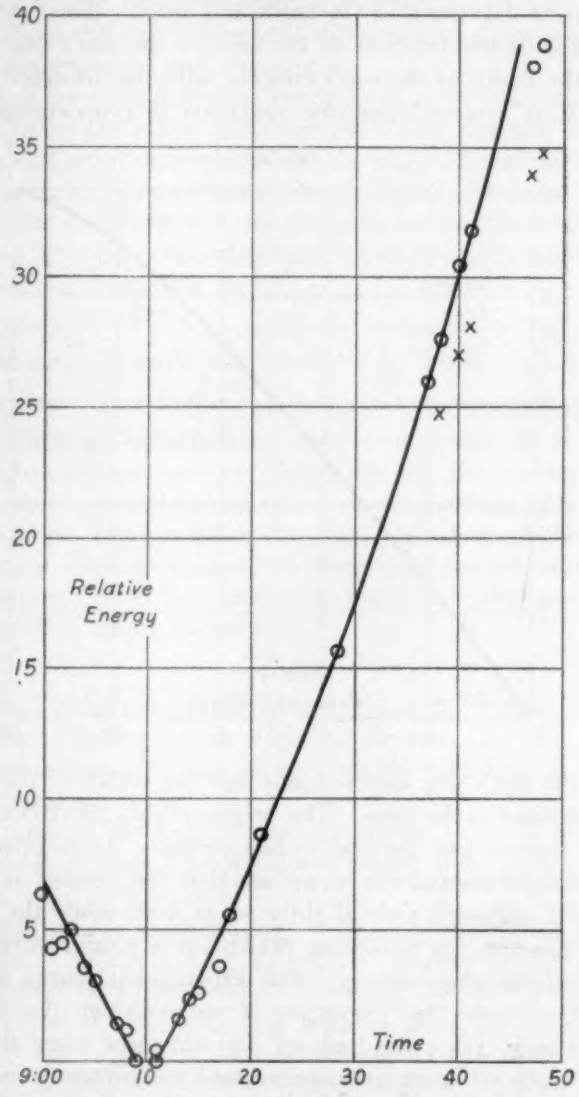
DISCUSSION OF RESULTS.

First contact occurred at 8:00 A.M. and at that time the prospect of being able to obtain radiation measurements at all seemed very doubtful. Galvanometer readings were occasionally taken, however, to get in practice in case the clouds should clear away. The fact that the sun at times could be seen through hazy clouds gave some hope. It is to be noted that the atmosphere in general was good. Shortly before 9:00 A.M. it was evident that some readings were to be obtained with a clear sky.

The curve, Fig. 1, shows the plot of galvanometer deflections against the time, beginning at 9:00 A.M. The first four points show the effect of clouds, but at 9:03 A.M. the sky cleared and remained so until 9:46 A.M. The sector-shaped aperture of the pyrometer was set for approximately 1 per cent. transmission as compared with the cover's being entirely removed. At second contact the cover was removed and five readings were obtained during totality. These cannot be shown on the scale used, but their actual values in centimetres of scale deflection were .4, .9, .5, .4, .3, the mean of which is .5. The ordinate scale used multiplied by 100 is centimetres of scale deflection corrected for aperture transmission and the galvanometer shunt. The pyrometer case was closed immediately after third contact to about 1 per cent. transmission. The fifth observation after third contact was probably erroneous, due to the instrument's getting "off the sun." There were no clouds interfering at that time. The ordinates of the last two observations are low, due to clouds, and the work was discontinued since the clouds persisted. The galvanometer shunt ratio was changed prior to the last five readings and when these were multiplied by the ratio (as computed and experimentally determined) the values were as indicated by the X's. The cause of this discrepancy is unknown, but in some way the sector aperture might have been jarred so as to change the transmission, although this suggestion is not based on any clew but on mere possibility. There were no clouds to cause the difference. Since the shunt was changed to keep the deflections on the scale, it seems obvious that the next reading, when corrected, would have been greater than the preceding. The ordinate of the first of the five points was therefore multiplied by a number which would line it up with the rest of the curve and then the other four multiplied by the same number.

A point to be considered when using a galvanometer to measure a varying current is the "overshooting" and "lagging."

FIG. 1.

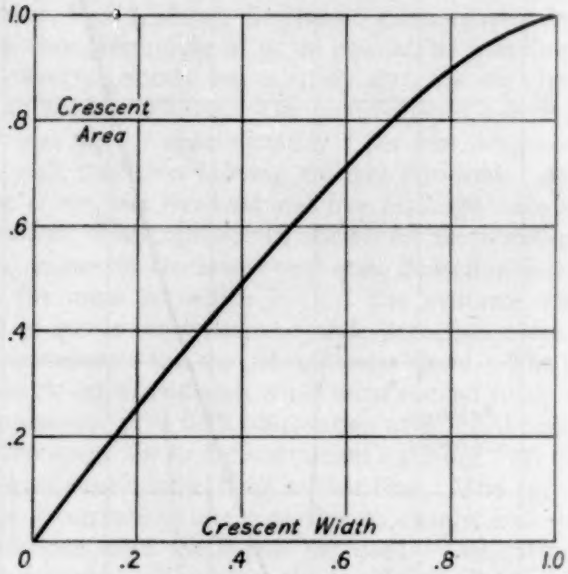


Computations with this in view were made in the present work, but it was found unnecessary to apply the correction as it

amounted at most to about .2 per cent. The author is indebted to Mr. L. Behr for assistance in computing.

Fig. 2 is a curve which was computed on the assumption that the apparent diameters of the sun's and moon's discs were equal. The plot is of the fraction of the area of the sun's disc exposed against the width of the sun's crescent with the diameter taken as unity. It is assumed that the radiation is proportional to the

FIG. 2.



area of the sun's disc exposed and that the width of the crescent is proportional to the time. The origin of this curve corresponds to third contact and the end to last contact. In comparing this with the experimental curve we see that the former is concave downward, although only slightly so at first, while the latter is concave upward, but becoming practically a straight line toward the end of the observations. The experimental curve would be expected to have this curvature if we consider the fact that radiant energy from the limb of the sun must come through a greater depth of solar atmosphere and consequently experience greater absorption and scattering than that from the centre of the disc. This, however, is not the entire explanation according to

Julius,[1] who says that an atmosphere on the sun is not *alone* sufficient to account for the variation.

If the theoretical curve had been computed on the basis of the true ratio of the moon's apparent diameter to that of the sun (1.02), about the only appreciable difference in shape would have been a different slope near the origin.

To find the scale deflection corresponding to full sunlight and thus determine the ratio of the energy of full sunlight to that of the corona, a point was selected on the experimental curve at the time 9:42 A.M. The ordinate here is 3300. At 9:42 A.M. the width of the sun's crescent, on the basis of the diameter's being unity, was .41. The computed curve gives the corresponding part of the sun's disc exposed as .51.

Therefore $\frac{.51}{1} = \frac{3300}{x}$, $X = 6500$, the ordinate for full sunlight. Since the mean ordinate for totality was .5 the desired ratio is $\frac{6500}{.5} = 13{,}000$. Observations on full sunlight January 30th gave a mean corrected deflection of 4600 centimetres or a ratio of 9200. Changed atmospheric conditions and the variability[2] of the sun would probably account for this difference, although the same error as introduced in the last five observations on the experimental curve is believed to have been carried over to this determination. If it has been introduced here the 9200 should be increased by about 12 per cent.

SUMMARY.

A Fery radiation pyrometer was used in relative radiation measurements during a total eclipse of the sun.

The experimental curve shows that the radiation is not uniform over the sun's disc.

The ratio of the energy received during full sunlight to the energy received from the corona was found to be 13,000.

Self-diffusion in Solid Metals. G. HEVESY and A. OBRUTSHEVA, Copenhagen. (*Nature*, May 9, 1925.)—Thanks to the work of Roberts-Austen and Stansfield we know not only that gold diffuses

[1] W. H. JULIUS, "The Total Solar Radiation during the Annular Eclipse on April 17, 1912," *Astrophys. Jour.*, **37**, 1913, p. 225, and "Interpretation of Photospheric Phenomena," *Astrophys. Jour.*, **38**, 1913, p. 129.

[2] C. G. ABBOT, F. E. FOWLE and L. B. ALDRICH, "On the Temperature and Radiation of the Sun," *Astrophys. Jour.*, 1916, **44**, p. 39.

into lead put in contact with it, but also at what rate it diffuses. At 100° C. this diffusion coefficient is 2×10^{-5} cm.^{2} day^{-1}. Langmuir has recently reported the diffusion of thorium into heated tungsten wires. There is no more direct proof of molecular motion in solids than that furnished by such instances of diffusion.

To study the diffusion of a metal into itself, that is, the rate at which the molecules depart from their previous positions and wander among their fellows of the same metal, is not at all easy. One cannot catch a molecule and band it as students of bird migration do with gulls and cormorants. This paper presents a developed form of a solution already applied by one of the authors. It depends on using isotopes of lead. These isotopes are so nearly of the same atomic mass that they diffuse at almost the same rate, but one of them emits α-rays and can thus be distinguished. Two pieces of thin foil were pressed together; one consisted of ordinary lead, the other of lead with radio-active thorium B uniformly distributed in it. The foil of ordinary lead was of such thickness that no α-particles from the radio-active foil could penetrate it. As time passed molecules of thorium B diffused into the other foil, and α-particles, that now came through, produced scintillations on a screen. From a comparison of the number of these with the number caused by the radio-active foil alone, it was possible to calculate the rate of diffusion of thorium B into ordinary lead. The coefficients show a rapid increase with temperature. Expressed in cm.^{2} day^{-1} units the values at 260°, 320° and 324° C. are, respectively, 6×10^{-7}, 4.7×10^{-5}, and $1.4 \times$ [illegible]. At the last-named temperature "the diffusion rate 2° below the melting-point is thus only 10,000 times smaller than in molten lead." Since in crystals the atoms are supposed to occupy definite positions on the space lattices, diffusion cannot be assumed to proceed as in non-crystalline matter. Thorium B was deposited on the surface of a single lead crystal and a decrease of scintillations, due to diffusion of the active material into the crystal, was sought but not found, even close to the melting-point. If diffusion occurred at all, it had a coefficient less than 10^{-8} cm.^{2} day^{-1}. In a lead bar, got by slow cooling of the molten lead, the coefficient was only twice that just given, while in a bar obtained by rapid cooling the coefficient was 10^{-5}. "The results found indicate that even the slow rate of diffusion observed just below the melting-point is not due to an exchange of place in crystals of appreciable size, but in the 'amorphous' material, which is found between the crystals and must necessarily show a less regular structure than the material composing the individual crystals, and thus will be more capable of allowing an exchange in the position of neighboring atoms. In a single lead crystal, or in a slowly cooled lead bar even only a few degrees below the melting-point, it would take longer, possibly very appreciably longer, than twenty years before an average displacement of the lead atoms to a distance of 1 cm. could take place. The time would amount to many million years at room temperature." G. F. S.

AN ANALOGY BETWEEN PURE MATHEMATICS AND THE OPERATIONAL MATHEMATICS OF HEAVISIDE BY MEANS OF THE THEORY OF H-FUNCTIONS.*†

BY

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The question of the continuity of the independent variable cannot be too strongly stressed. It constitutes an essential difference between the theory of H-operations as given here and the Theory of Functions of a Real Variable. We cannot make any use of this property of continuity of the variable x if we simply name definite values of x. Any two values of x we name, no matter how nearly equal, have an infinite number of other values between them. In order to use the property of continuity of x, we must let x not only have a certain value, but we must also let it approach this value. The only way in which it can do this is through a limiting process. Furthermore in the Theory of H-functions, no limiting process has precedence over another. It will thus be noticed that in summing (42) in the form (43) we have allowed the limiting processes with respect to x and n to go on concurrently. In the rule given in the Theory of Functions of a Real Variable, the limiting process with respect to x may be regarded as taking precedence over that with respect to n, since the first thing that is done is to insert the actual value of x for which the series is to be summed, *i.e.*, x may be regarded as having approached its limit first and then the limit of the sum is taken as n approaches infinity.

It is very important to remember that in the Theory of H-functions all limiting processes must go on concurrently. For instance, it is not unusual to find in the Theory of Functions of a Real Variable examples where

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x,y) \tag{45}$$

is not equal to

$$\lim_{y \to y_0} \lim_{x \to x_0} f(x,y) \tag{46}$$

There is no reason why they should be equal. In fact, in cases

* Communicated by the Author.

† Continued from p. 534 of the October issue.

where it is true it would appear to be a happy accident rather than anything else. Now in cases where it is required to find the limits, it will generally be found that what is required is neither (45) nor (46), but rather the value of $f(x, y)$ as $x \to x_0$ and $y \to y_0$ with no reference to any priority of one of these limiting processes over the other. It will be found that if x and y are allowed to approach their limits simultaneously, we can obtain every value between those given by (45) and (46) and perhaps other values in addition.

To illustrate we may perhaps go back to the problem of finding the sum of the series (42) in the neighborhood of the origin. In (43) we have its sum replaced by

$$y_1' \longleftrightarrow \frac{1}{\pi}\int_0^{\frac{2n}{p}} \frac{\sin\alpha}{\alpha}\, d\alpha \tag{43}$$

where we have not yet named our value of x (or p) and n. If we now let the limiting process with respect to n take place first, we get

$$y_1' \longleftrightarrow \frac{1}{\pi}\int_0^{\infty} \frac{\sin\alpha}{\alpha}\, d\alpha \longleftrightarrow \frac{1}{2} \tag{43a}$$

If we let the limiting process with respect to x take place first, we get

$$y_1' \longleftrightarrow \frac{1}{\pi}\int_0^{0} \frac{\sin\alpha}{\alpha}\, d\alpha \longleftrightarrow 0 \tag{43b}$$

But by letting both limiting processes go on simultaneously, we get all values from 0 to $\frac{1}{2}$ and other values in addition.

Thus the H-sum of an infinite series is defined to be

$$\operatorname*{Lim}_{\substack{x \to x_0 \\ n \to \infty}} S_n(x) \tag{47}$$

where Lim denotes x and n are to approach their limits simultaneously, and this definition is to be carefully distinguished from the sum of an infinite series defined in the Theory of Functions of a Real Variable as

$$\operatorname*{Lim}_{n \to \infty} [\operatorname*{Lim}_{x \to x_0} S_n(x)] \tag{48}$$

or as (48) might perhaps be more correctly written

$$\operatorname*{Lim}_{n \to \infty} S_n(x_0) \tag{48a}$$

where x_0 is a definitely named number.

As a general rule it will be found that the application of (47) is a more difficult matter than the application of (48). We will now endeavor to show how the theory of H-uniform convergence assists us in evaluating (47).

The definition of uniform convergence from the Theory of Functions of a Real Variable is: When the series

$$u_1(x) + u_2(x) + \ldots\ldots + u_n(x) + \ldots.$$

converges for all values of x in the interval $a \leq x \leq b$ to the value $f(x)$, it is said to be uniformly convergent in this interval when a positive quantity ε being chosen as small as we please, a finite positive integer ν exists such that for all values of x in the interval

$$|f(x) - S_n(x)| < \varepsilon \quad \text{for} \quad n \geq \nu$$

in other words

$$|R_n(x)| < \varepsilon \quad \text{for} \quad n \geq \nu$$

for all values of x in the interval.

This definition, provided it is interpreted properly, gives a criterion for the interchangeability of (47) and (48). The most important phrase in the definition is " for all values of x in the interval." Our postulate for an H-function requires that the domain of x be continuous. It is obvious that we cannot hope to make use of the full meaning of this definition of uniform convergence by simply inserting values of x in the expression for $R_n(x)$. We must, as in the case of the H-sum of a series, allow x not only to take on any value in the interval, but it must be allowed to approach it in a perfectly arbitrary manner. As has been pointed out before, to name definitely any value of x first and then take the limit with respect to n is to allow the limiting process with respect to x definite precedence over that with respect to n.

Let us consider how this test might be applied to a series (which we will take from Carslaw, " Fourier Series," 1921, pp. 126 and 127).

Consider the series

$$u_1(x) + u_2(x) + \ldots. \qquad x \gtreqless 0 \tag{49}$$

where

$$u_n(x) = \frac{nx}{1 + n^2x^2} - \frac{(n-1)x}{1 + (n-1)^2x^2} \tag{50}$$

In this case

$$S_n(x) = \frac{nx}{1 + n^2x^2} \tag{51}$$

and

$$\operatorname*{Lt}_{n\to\infty}\ [\operatorname*{Lt}_{x\to x_0} S_n(x)] \longleftrightarrow 0 \text{ for all values of } x \tag{52}$$

We will take this latter function $f(x) = 0$ for all values of x to be the function $f(x)$ in the definition of uniform convergence. Then the value of $R_n(x)$ becomes

$$R_n(x) = \frac{nx}{1 + n^2x^2} \tag{53}$$

In order to be able to find the various possible values of $R_n(x)$ as n and x vary independently, we must put this expression into such a form that its value depends only upon one arbitrary constant. Let us assume the value is $f(k)$, which is some function of the arbitrary constant k. Then

$$\frac{nx}{1 + n^2x^2} = f(k) \tag{53a}$$

or

$$x = \frac{1}{2nf(k)} \pm \sqrt{\frac{1}{4n^2f^2(k)} - \frac{1}{n^2}} \tag{54}$$

Since the domain of x is real, the quantity under the radical must be positive, hence

$$\frac{1}{4n^2f^2(k)} - \frac{1}{n^2} \geq 0 \tag{55}$$

giving

$$-\frac{1}{2} \leq f(k) \leq +\frac{1}{2} \tag{56}$$

This gives the maximum and minimum values of $f(k)$ as $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and for these values $x = \frac{1}{n}$ and $-\frac{1}{n}$, respectively. It should be carefully noted that so far no particular value of x has been named. Now let x have any value p/n, so that x still remains unnamed on account of the arbitrary constant p, then

$$R_n(x) \longleftrightarrow \frac{p}{1 + p^2} \qquad \left(x = \frac{p}{n}\right) \tag{57}$$

and by giving different values to p we now find values for $R_n(x)$ ranging continuously from $-\frac{1}{2}$ to $+\frac{1}{2}$, the limits already obtained above.

We have thus far only proved that in the interval $x \gtreqless 0$ the maximum value of $R_n(x)$ is $\pm\frac{1}{2}$ and that this value occurs as x approaches zero. We will now show that as we approach any

definite value of x not equal to zero, the value of $R_n(x)$ can be made less than any quantity ε we choose to name. To fix our ideas let us consider the series in the neighborhood of the point x_1. Then we find as before, when Δx is a small unnamed increment in x, that

$$x_1 + \Delta x = \frac{1}{2nf(k)} \pm \sqrt{\frac{1}{4n^2f^2(k)} - \frac{1}{n^2}} \tag{58}$$

Let us assume that x_1 has been chosen as an essentially positive number, and let the limits of the neighborhood in which we desire to consider the value of the series be $x_1 + \Delta x_1$ and $x_1 - \Delta x_2$, where Δx_1 and Δx_2 are definitely named positive numbers, and furthermore $x - \Delta x_2 > 0$, so that our interval does not include the point $x = 0$. Then the maximum value of the left-hand side of (58) is $x_1 + \Delta x_1$, and the minimum value is $x_1 - \Delta x_2$. Hence if we assume the square root is taken with the positive sign

$$x_1 + \Delta x_1 > \frac{1}{2nf(k)} + \sqrt{\frac{1}{4n^2f^2(k)} - \frac{1}{n^2}} \tag{59}$$

and

$$x_1 - \Delta x_2 < \frac{1}{2nf(k)} - \sqrt{\frac{1}{4n^2f^2(k)} - \frac{1}{n^2}} \tag{60}$$

From (59) it is easily found that

$$f(k) > \frac{1}{n}\left[\frac{x_1 + \Delta x_1}{(x_1 + \Delta x_1)^2 + \frac{1}{n^2}}\right] \tag{61}$$

and from (60)

$$f(k) < \frac{1}{n}\left[\frac{x_1 - \Delta x_2}{(x_1 - \Delta x_2)^2 + \frac{1}{n^2}}\right] \tag{62}$$

In (61) and (62) x_1, Δx_1, Δx_2 have definite values such that $(x_1 + \Delta x_1)$ and $(x_1 - \Delta x_2)$ are positive. Let

$$x_1 - \Delta x_2 = p \qquad 0 < p$$
$$x_1 + \Delta x_1 = q \qquad 0 < q$$

Then we have shown that as x lies in the continuous interval $p < x < q$

$$f(k) < \frac{p}{p^2 + \frac{1}{n^2}} \cdot \frac{1}{n} \tag{63}$$

and

$$f(k) > \frac{q}{q^2 + \frac{1}{n^2}} \cdot \frac{1}{n} \tag{64}$$

Then from (63)

$$\operatorname*{Lim}_{\substack{p<x<q \\ n\to\infty}} f(k) \longleftrightarrow 0 \tag{65}$$

or

$$\operatorname*{Lim}_{\substack{x\to x_1 \\ n\to\infty}} R_n(x) \longleftrightarrow 0 \tag{66}$$

since x_1 is any point such that $p < x_1 < q$.

Hence the H-sum of the series in the neighborhood of any positive definitely named value of x, say x_1, $(x_1 \neq 0)$ converges to zero. Similarly it can be shown that the H-sum of the series in the neighborhood of any negative value of x, say x_2, $(x_2 \neq 0)$ converges to zero.

We may state what we have proved in another way as follows: If

$$y = u_1(x) + u_2(x) + \cdots\cdots \tag{67}$$

where

$$u_n(x) = \frac{nx}{1 + n^2x^2} - \frac{(n-1)x}{1 + (n-1)^2x^2}$$

then

$$y \longleftrightarrow f(x) \qquad x \gtrless 0 \tag{68}$$

where

and

$$\left.\begin{aligned} f(x) &= 0 \qquad x \neq 0 \\ f(x) &= \frac{k}{1 + k^2} \qquad x = 0 \end{aligned}\right\} \tag{69}$$

The above method of showing that in the neighborhood of a point $x_1 (x_1 \neq 0)$ the sum of the series converges to zero, would doubtless be very difficult of application to series in general. To deal with other cases it would be desirable to have other tests for H-uniform convergence, similar to the alternative tests for uniform convergence used in the Theory of Functions. For instance, let us consider an alternative H-M-test equivalent to Weierstrass's M-test for uniform convergence as follows:

The series

$$u_1(x) + u_2(x) + \cdots\cdots \tag{70}$$

will converge uniformly in the neighborhood of the point x_1, if there exists a series of positive constants

$$M_1 + M_2 + M_3 + \cdots\cdots \tag{71}$$

such that whatever value x may have in the neighborhood of x_1

$$|u_n(x)| < M_n \qquad n = 1, 2, 3, \text{ etc.}$$

provided the M series is convergent.

Since the M series converges

$$R_n(M_n) < \varepsilon \qquad n \geq \nu$$

But every term in the series $u_n(x)$ is less than the corresponding term in the M series, hence

$$|R_n[u_n(x)]| < \varepsilon$$

If we can thus prove without naming a definite value of x that $|R_n[u_n(x)]|$ is less than ε for every point in the neighborhood of x_1, we can obtain the sum of the series in the neighborhood of the point by inserting definite values of x in the original series.

For instance, consider the series

$$y = 1 - x + x^2 - x^3 + \ldots\ldots \tag{72}$$

and let its sum be required in the neighborhood of the point $x = \frac{1}{2}$. Let us take the neighborhood of the point to be the interval $\frac{1}{4} < x < \frac{3}{4}$. Then for any point in this interval a suitable series for M will be

$$M = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots\ldots \tag{73}$$

so that

$$|R_n(x)| < \frac{\left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}} \tag{74}$$

Then having decided upon a value of ε we can determine ν such that for $n \geq \nu$

$$\varepsilon > R_n(x) > -\varepsilon \tag{75}$$

Thus no matter how x and n approach their limits in the neighborhood of the point $x = \frac{1}{2}$, if $n \geq \nu$ the maximum and minimum values of $R_n(x)$ are $+\varepsilon$ and $-\varepsilon$, respectively. To find the sum of the series, therefore, we let x approach its limit first, *i.e.*, insert the actual value of x in the series and then sum it as a geometrical progression in real numbers. The value thus obtained differs by less than ε from the value which would be obtained if the sum

were taken by allowing x and n to approach their limits simultaneously in a perfectly arbitrary manner.

We might now endeavor to develop other equivalent tests for H-uniform convergence, and to show the application to other series. As our purpose, however, is to attempt to outline a theory under which many of the operations used by Heaviside might be made legitimate, we will not attempt to develop further the theory of H-uniform convergence. Some remarks may nevertheless be added in order to emphasize the difference between the theory given here and the theory as developed in the Theory of Functions of a Real Variable.

The theory of convergence has no place in the Theory of H-functions. Convergence only applies to a series of pure numbers, as, for instance, we may say the series

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots \tag{76}$$

is convergent, but as pointed out above, this is not equivalent to saying that the H-sum of the series

$$1+x+x^2+\cdots \tag{77}$$

is H-convergent in the neighborhood of the point $x=\frac{1}{2}$. The theory of H-uniform convergence is necessary to prove the latter statement. A good example to show how the ordinary theory of convergence fails when dealing with H-functions may be taken from Carslaw, "Fourier Series," p. 127 (1921).

Consider the series

$$u_1(x)+u_2(x)+\cdots \qquad x \gtreqless 0 \tag{78}$$

where

$$u_n(x)=\frac{n^2x}{1+n^2x^2}-\frac{(n-1)^2x}{1+(n-1)^2x^2} \tag{79}$$

when the value $x=0$ is inserted in this series all the terms are zero, and the limit of their sum is zero. Hence the series is convergent at this point according to the Theory of Functions of a Real Variable. If, however, we take the H-sum, we easily find that by letting x approach zero by taking $x=\frac{k}{\sqrt{n^3}}$, we get for the H-sum of the series

$$\frac{\sqrt{n}\,k}{1+k^2} \tag{80}$$

which varies from 0 to ∞ as k varies from 0 to ∞. Thus the H-sum is not convergent.

We may explain the difference as follows, by saying that the sum of the number series

$$0+0+0+0+\cdots \tag{81}$$

approaches the limit zero, but the H-sum of the series $u_n(x)$ cannot be obtained by inserting the value $x=0$ alone, but must be taken in the neighborhood of the point $x=0$ when of course we get all the various values given by (80). Incidentally (80) contains the value 0 obtained by the method of the Theory of Functions in addition to an infinite number of other values. In the theory of H-functions it is assumed that the H-sum of a series obtained by letting x and n approach their limits independently can have an infinite number of different values, until we prove that in the neighborhood of any particular point these values all differ from each other by a quantity less than ε. If we cannot prove that such a quantity ε exists, then the H-sum at the point may have any of the values which can be obtained by letting x and n approach their limits independently or simultaneously.

Another point which may be emphasized is the fact that we do not write

$$\frac{1}{1-x} = 1+x+x^2+\cdots\cdots \qquad (-1<x<1) \tag{82}$$

Let us examine what we can prove for any point in the interval $(-1<x<1)$ similar to what we proved for the point $x=\frac{1}{2}$ above. We can show that for any point in this interval the difference between the H-sum of the series and the fraction $\frac{1}{1-x}$ lies in the range $-\varepsilon<R_n(x)<\varepsilon\,(n\geq\nu)$ where having named ε, we can determine a value of ν. But the whole proof is conditional upon our naming a definite value of ε. There seems to be some confusion on this point in the present Theory of Functions of a Real Variable, when an equation such as (82) is written down. It is said there, of course, that ε may be made as small as we please, and when we write the equality sign between the two sides of this relation, there seems at least to be a tacit assumption that ε can be equal to zero. But if we let ε equal to zero the whole proof breaks down. On the other hand, it may also be argued that this is a mere matter of definition of equality. If this be granted,

then it is obvious that we use the equality sign under two different interpretations. For instance, when we write

$$(x+a)^2 = x^2 + 2ax + a^2 \tag{83}$$

we have a relation, to prove which, it is not necessary to name any quantity ε such that if the difference between the quantities on either side of this equation is less than ε, we will agree to call them equal. In (82) we do, on the contrary, have to name an ε. For this reason it has seemed best in developing the theory of H-functions to use the equality sign in equations like (83), where no ε is required in the proof, and to use a $\longleftrightarrow$ sign in equations like (82), where an ε is required in the proof. Thus we shall write

$$\frac{1}{1-x} \overset{(-1<x<1)}{\longleftrightarrow} 1 + x + x^2 + \cdots \tag{84}$$

where the $\overset{(-1<x<1)}{\longleftrightarrow}$ sign is to be read that $1/(1-x)$ converges to the H-sum of the series $1 + x + x^2 + \ldots$ in the interval $(-1 < x < 1)$ and *vice versa*. This would appear to be a complete statement of all that we can actually prove.

It should now be obvious that such a notation gives an additional reason for defining H-integration as the inverse process of H-differentiation. For suppose, to take a simple example, we take the H-differential coefficient of the H-function

$$y = x^2 \tag{85}$$

we get

$$\frac{Dy}{Dx} = 2x \tag{86}$$

Now if we attempt to use the Riemann or Lebesgue or any other of the definitions given in the Theory of Functions of a Real Variable, we shall find that even if we do our best to interpret them in the broadest sense possible, all we can prove is that

$$x^2 - \varepsilon < \int_0^x 2x dx < x^2 + \varepsilon \tag{87}$$

so we can only say that $\int_0^x 2x dx$ converges to x^2, and write it

$$\int_0^x 2x dx \overset{x \geqq 0}{\longleftrightarrow} x^2 \tag{88}$$

Our definition of H-integration, however, defines the H-integration process to be such that if

$$\frac{Dy}{Dx} = f_P(x) \tag{89}$$

then

$$\int f_P(x)\,dx = y + C \tag{90}$$

We can, however, make use of the Riemann, Lebesgue and other definitions of integrals as approximate methods for determining an H-integral, provided the test of H-uniform convergence is applied to the sum of the infinitesimals which go to make up these integrals. This interpretation gives a reasonable explanation of what may otherwise prove a very puzzling question in the Theory of Functions of a Real Variable. If we were to ask in the Theory of Functions, "What is an integral?", it would be rather difficult to reply. If we take Hobson's "Theory of Functions," second edition, Vol. i, we find Riemann integrals defined. We go further in the volume and find Lebesgue integrals, Stieltjes integrals, Denjoy integrals, Harnack integrals and then we begin to get combinations as, for instance, the Denjoy-Khintchine-Young integral (D-K-Y integral). One may well ask, "What is an integral?" Now from the viewpoint of the theory of H-functions, the process of thus defining integrals may well go on for ever, since there is an infinite number of ways of dividing up the interval over which the integral is to be taken.

As a matter of fact it does not appear that any of these definitions will be of use in connection with the integration of the second H-derivative of the H-function

$$\begin{aligned} y &= (1-\xi)\,x && x \leq \xi \\ y &= (1-x)\,\xi && x \geq \xi \end{aligned} \tag{4}$$

unless some other data are given as to its behavior at the point $x = \xi$.

One more point may be added in connection with the series already discussed in (38)

$$y = \frac{2}{\pi^2}\left[\sin \pi x \sin \pi\xi + \frac{\sin 3\pi x \sin 3\pi\xi}{3^2} + \cdots\right] \tag{38}$$

This series may be shown by the H-M-test to be H uniformly convergent and thus its H-sum converges in the neighborhood of any point x_1 to the value obtained by inserting the actual value x_1 in the series and then summing. If this is done we can show, as in the Theory of Functions, that the H-sum or ordinary sum

thus obtained (since they are equivalent) converges to the function given by

$$f(x) = (1-\xi)\frac{x^2}{2} \qquad x < \xi$$
$$f(x) = \left(1 - \frac{x^2}{2}\right)\xi \qquad x > \xi \tag{91}$$

But in the theory of H-functions, we should then write

$$y \overset{0<x<1}{\longleftrightarrow} f(x) \tag{92}$$

or we may write it

$$|f(x)| - \varepsilon_1 < |y| < |f(x)| + \varepsilon_2 \tag{92a}$$

Let us now consider the H-functions $\frac{Dy}{Dx}$ and $\frac{Df(x)}{Dx}$ where $\frac{Dy}{Dx}$ means the series obtained by differentiating the series for y term by term. If this series is H uniformly convergent in a certain interval, which we will call (a, b), then there exists a function $\phi(x)$ such that if $\phi(x)$ and $\frac{Dy}{Dx}$ are positive in the interval (if they are not the proof is easily modified)

$$\phi(x) - \eta_1 < \frac{Dy}{Dx} < \phi(x) + \eta_2 \tag{93}$$

Applying the Riemann integral to this inequality

$$\int_{x_0}^{x_1} \phi(x)dx - \int_{x_0}^{x_1} \eta_1 dx < y_1 - y_0 < \int_{x_0}^{x_1} \phi(x)dx + \int_{x_0}^{x_1} \eta_2 dx \tag{94}$$

by choosing η_1 and η_2 so that $\int_{x_0}^{x_1} \eta_1 dx < \varepsilon_1$ and $\int_{x_0}^{x_1} \eta_2 dx < \varepsilon_2$, we have

$$\int_{x_0}^{x_1} \phi(x)dx - \varepsilon_1 < y_1 - y_0 < \int_{x_0}^{x_1} \phi(x)dx + \varepsilon_2 \tag{95}$$

hence we can conclude

$$\int_{x_0}^{x_1} \phi(x)dx \overset{a<x<b}{\longleftrightarrow} f(x_1) - f(x_0) \tag{96}$$

or

$$f(x_1) - f(x_0) - \varepsilon_3 < \int_{x_0}^{x_1} \phi(x)dx < f(x_1) - f(x_0) + \varepsilon_4$$

by suitably choosing ε_3 and ε_4. Now by the theorem of mean value (we have assumed here that $\phi(x)$ is a function expressed in finite form with no other limiting process except the one with respect to x_1, and furthermore that $\phi(x)$ is continuous and has only one definite value at every point in the interval, so that the

methods of the Theory of Functions of a Real Variable are available) if we let $x_0 = x - \Delta x_2$ and $x_1 = x + \Delta x_1$, then

$$\phi(\xi)dx \longleftrightarrow \int_{x-\Delta x_2}^{x+\Delta x_1} \phi(x)dx \tag{97}$$

where ξ is some value of x in the interval $(x_1 - \Delta x_2 < x < x_1 + \Delta x_1)$. Hence from (96)

$$\begin{aligned} \phi(\xi) &\longleftrightarrow \frac{f(x+\Delta x_1) - f(x-\Delta x_2)}{\Delta x_1 + \Delta x_2} \\ &\longleftrightarrow \frac{Df(x)}{Dx} \end{aligned} \tag{98}$$

or, if a series converges to a definite function $f(x)$ and the series obtained by differentiating the original series term by term converges H uniformly to a function $\phi(x)$, then this function $\phi(x)$ converges to the H-differential coefficient of $f(x)$.

There is a further point in connection with the theory of H-uniform convergence which may be pointed out here. Another definition of H-uniform convergence may be adopted as follows: If the series

$$u_1(x) + u_2(x) + u_3(x) + \cdots\cdots$$

is considered as an H-sum in the interval $a \leq x \leq b$, it is said to be H uniformly convergent in this interval when, a positive quantity ε being chosen as small as we please, a finite positive integer ν exists such that for all values of x in the interval

$$|S_{n+p}(x) - S_n(x)| < \varepsilon \quad \text{for } n \geq \nu \text{ and } p = 1, 2, 3, \cdots$$

in other words

$${}_pR_n(x) < \varepsilon \qquad n \geq \nu \text{ and } p = 1, 2, 3, \cdots$$

It will be noted that this definition of H-uniform convergence is entirely independent of any function $f(x)$ such as is required in the former definition. It would appear as if this definition, which does not require a consideration of any function $f(x)$, is the true definition of H-uniform convergence. It was not introduced at first because it is slightly more difficult to apply than the other definition. Consider for a moment its application to the series already used in (49)

$$u_1(x) + u_2(x) + u_3(x) + \cdots \qquad x \gtrless 0 \tag{49}$$

where

$$u_n(x) = \frac{nx}{1 + n^2x^2} - \frac{(n-1)x}{1 + (n-1)^2x^2} \tag{50}$$

Then

$$ {}_pR_n(x) = \frac{(n+p)x}{1+(n+p)^2x^2} - \frac{nx}{1+n^2x^2} \qquad (99) $$

It can now be shown, as before, that both $\frac{(n+p)x}{1+(n+p)^2x^2}$ and $\frac{nx}{1+n^2x^2}$ have maximum and minimum values of $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and that these values occur as x approaches zero from the right and left, respectively. Further, it can be shown, as before, that in the neighborhood of any point x_0 $(x_0 \neq 0)$ the value of each of these fractions can be made to lie between $+\varepsilon$ and $-\varepsilon$ where ε can be made as small as we please. Hence we can write

$$ -2\varepsilon < {}_pR_n(x_0) < +2\varepsilon \qquad x_0 \neq 0 \qquad (100) $$

It can thus be shown that in the neighborhood of any point x_0 $(x_0 \neq 0)$ the H-sum of the series converges to a single value for $n \geq \nu$ (since we have placed no restriction on p except that it is positive and may be as large as we please). Furthermore, it has been shown that in the neighborhood of the point $x = 0$ the H-sum of the series lies between $+\frac{1}{2}$ and $-\frac{1}{2}$ (this is because if x is positive both fractions are positive and if x is negative both are negative). By letting $x = k/n$ we can easily prove in addition by varying p for any given value of k that the value of ${}_pR_n(x)$ can converge to any value between $+\frac{1}{2}$ and $-\frac{1}{2}$ in the neighborhood of the point $x = 0$.

This latter test will be called the test of H-1-uniform convergence. The former test where the function $f(x)$ is used will be called H-2-uniform convergence. There are objections to the H-2-test for uniform convergence which arise in connection with the definition of $f(x)$ itself. Up to this point it has been assumed, tacitly at least, that $f(x)$ is a function which has only one value at every point in its range. This is perfectly satisfactory as long as $f(x)$ is a continuous function, but it is not satisfactory when $f(x)$ is a so-called discontinuous function.

DEFINITION OF AN H-FUNCTION.

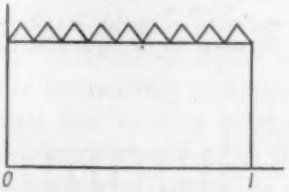
In the foregoing we have been fortunate in the fact that the ordinary mathematical definitions could be used with slight alterations in the theory of H-functions. The concepts of

H-function, H-limit and H-continuity present greater difficulty, and it will be noted in these cases that the ordinary mathematical definitions give little assistance.

Consider a very simple example of a function as ordinarily defined in the Theory of Functions of a Real Variable. Let x lie in the interval 0 to 1, and let the value of y be a constant unity corresponding to any value of x in this interval.

Let us compare this with the following example used by Hilbert for another purpose. Draw a line parallel to the x-axis

FIG. 1.



passing through the point $y = 1$. Take the interval on this line as x goes from 0 to 1, divide it up into n equal parts, and on each of these n equal parts describe an equilateral triangle. Let the curve formed by the bases of these triangles be the curve A, and the curve formed by the sides of the triangles the curve B. Then it is evident that the distance described going from $x = 0$ to $x = 1$ along the curve B is equal to 2, and it remains constant as n is increased indefinitely. If the word limit is used in the ordinary mathematical sense, the limit of the curve B as n is increased indefinitely, is the straight line $y = 1$ or the curve A, and the distance travelled by a point going from 0 to 1 along the curve A is unity.

In the theory of H-functions it is both important and necessary to distinguish between these two results. The H-limit to which the curve formed by the sides of the equilateral triangles approaches must be regarded as altogether different from the function $y = 1$, we will write

$$\underset{n \to \infty}{H\text{-limit}} \ (\text{curve } B) = \beta \tag{101}$$

where β is merely a symbol used to denote this H-limit. Also

$$\underset{n \to \infty}{H\text{-limit}} \ (\text{curve } A) = (\text{the curve } y = 1) = \alpha \tag{102}$$

and since it is possible to show that for any value of x the values

obtained by each of these methods differ numerically by less than ε, we may write

$$H\text{-limit}_{n\to\infty} \text{ (curve } B) \longleftrightarrow H\text{-limit}_{n\to\infty} \text{ (curve } A) \qquad (103)$$

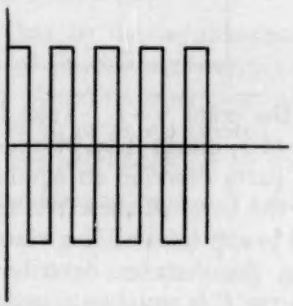
which denotes that numerically these limits converge to one another.

Similarly by describing semi-circles on the n intervals and increasing n indefinitely, another curve C would be obtained, which might be written in the form

$$H\text{-limit}_{n\to\infty} \text{ (curve } C) = \gamma \qquad (104)$$

where γ is again a symbol to denote the limit. Continuing this,

FIG. 2.



a whole series of other curves, δ, ε, η, might be obtained, and between them the following relations exist:

$$\alpha \longleftrightarrow \beta \longleftrightarrow \gamma \longleftrightarrow \delta, \text{ etc.} \qquad (105)$$

To return again to the curve B formed by the sides of the n equilateral triangles, if its H-differential coefficient is taken, it will be found to consist of vertical lines at $2n$ points and of constant values of $\pm\sqrt{3}$ in between (Fig. 2). The limit approached by the H-differential coefficient as n is increased indefinitely must then be regarded as any point in the space between $y = +\sqrt{3}$ and $y = -\sqrt{3}$. In other words, as we attempt to take the H-limit, any characteristic which exists as we approach the H-limit must be retained and assumed to exist in the H-limit itself.

The real difficulty which underlies the reversal of limiting processes in the Theory of Functions of a Real Variable should be apparent from the above example. It apparently must arise when the ε method of taking a limit is used. This is one of

the great disadvantages of the Cantor system of numbers frequently used as the foundation of the Theory of Functions. In the Cantor system, starting from the rational numbers which are perfectly definite and involve no limiting process, in order to extend the idea to irrational numbers a limiting process is introduced, and finally even the rational numbers which existed initially are discarded, and all numbers are defined by sequences which involve the definition of a limit by means of an ε. Thus even in the definition of a number itself there is a limit involved in the Cantor system. The Dedekind cut theory seems to offer greater possibilities of developing a number system free from this limiting process. However, even in the theory of H-functions, the Cantor system may possibly be used, provided that it is recollected that this limiting process is involved. In fact, in defining the H-sum of a series, the independent variable was made to take its successive values by means of a limiting process.

There is another point which may be brought out in connection with the limiting process as used in the Theory of Functions. This is that the ε limit is taken with respect to only one of the properties of the dependent variable. Thus in taking the ε limit of the curve B above, the only characteristic considered is the distance from the x-axis to a point on the curve B. However, another perfectly definite characteristic of the curve B is the distance along it as x goes from 0 to 1, and this characteristic approaches the limit $2x$. Other characteristics might also be taken, such as the directions travelled in going along the curve. Now it may happen that we are not interested in the distance from the x-axis to a point on the curve, but in some of the other characteristics. If the methods of the Theory of Functions of a Real Variable are employed, these characteristics are lost.

The following example may be given to show the difficulties thus introduced by taking the limit with respect to one characteristic alone. It is an example which will be useful in connection with the theory of Green's functions later, and it will be seen there that the property of the H-function which will be of interest at that time is, not the value of the H-function itself, but rather the behavior of the value of its H-integral at a certain point in the range of integration. The example is considered by Byerly,

"Fourier Series," pp. 55–60. We shall consider it here from a slightly different aspect.

Byerly considers the integral of the series

$$y = \sin x \sin \alpha + \sin 2x \sin 2\alpha + \cdots \tag{106}$$

and shows that if y_n is the sum of the first n terms of the series

$$\operatorname*{Lim}_{n \to \infty} \int_0^{\frac{\pi}{2}} y_n d\alpha = \frac{\pi}{2} \tag{107}$$

where the limit implied in the integral sign is to be taken first, and then the limit taken as $n \to \infty$. He carefully points out that these limiting processes cannot be reversed in order to obtain the more important result

$$\int_0^{\frac{\pi}{2}} y d\alpha = \int_0^{\frac{\pi}{2}} \operatorname*{Limit}_{n \to \infty} y_n d\alpha = \frac{\pi}{2}. \tag{108}$$

Considered, however, in the theory of H-functions, neither limit has priority over the other, and the latter result is true. In fact, whatever limiting processes are involved, they must be taken so that it is true, just as with the curve B above, the limiting process has to be taken so that the distance measured along the curve is $2x$.

Let us now consider the value of

$$\int_0^b y d\alpha.$$

To do so we shall make use of Dirichlet's integrals in the second form. These will be found in Carslaw, "Fourier Series," second edition, page 207, or in other works on this subject. They are

$$\operatorname*{Limit}_{n \to \infty} \int_0^a f(x) \frac{\sin nx}{\sin x} dx \longleftrightarrow \frac{\pi}{2} f(+0) \tag{109}$$

$$\operatorname*{Limit}_{n \to \infty} \int_a^b f(x) \frac{\sin nx}{\sin x} dx \longleftrightarrow 0 \tag{110}$$

subject to certain restrictions on $f(x)$ and where $0 < a < b < \pi$.

If $0 < x < a < \pi$ the integral of the first n terms of the series (106) may be written as shown by Byerly,

$$\int_0^a y_n d\alpha = \frac{1}{2} \int_{-\frac{x}{2}}^{\frac{a}{2} - \frac{x}{2}} \frac{\sin (2n+1)\beta}{\sin \beta} d\beta - \frac{1}{2} \int_{\frac{x}{2}}^{\frac{a}{2} + \frac{x}{2}} \frac{\sin (2n+1)\beta}{\sin \beta} d\beta \tag{111}$$

breaking up the first integral and applying some simple transformations

$$\int_0^a y_n d\alpha = \frac{1}{2}\int_0^{\frac{a}{2}-\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta + \frac{1}{2}\int_0^{\frac{x}{2}} \frac{\sin(2n+1)\gamma}{\sin\gamma}\,d\gamma - \frac{1}{2}\int_{\frac{x}{2}}^{\frac{a}{2}+\frac{x}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta. \tag{112}$$

Taking the limit as $n \to \infty$, the first two integrals give $\pi/2$ and the third zero, hence

$$\int_0^a y d\alpha \longleftrightarrow \frac{\pi}{2} \qquad 0 < x < a < \pi \tag{113}$$

If, however, $0 < a < x < \pi$

$$\int_0^a y_n d\alpha = \frac{1}{2}\int_{\frac{x}{2}}^{\frac{x}{2}-\frac{a}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta - \frac{1}{2}\int_{\frac{x}{2}}^{\frac{x}{2}+\frac{a}{2}} \frac{\sin(2n+1)\beta}{\sin\beta}\,d\beta \tag{114}$$

and when the limit is taken as $n \to \infty$, both the integrals on the right-hand side are zero from (110) and thus

$$\int_0^a y d\alpha \longleftrightarrow 0 \qquad 0 < a < x < \pi \tag{115}$$

Hence

$$\int_0^a y d\alpha \longleftrightarrow \frac{\pi}{2} \qquad 0 < x < a < \pi \tag{116}$$

$$\longleftrightarrow 0 \qquad 0 < a < x < \pi \tag{117}$$

This result may also be written in the form in which it will have useful application in connection with the theory of Green's functions

$$\int_{x-\varepsilon}^{x+\varepsilon} y d\alpha \longleftrightarrow \frac{\pi}{2} \qquad 0 < x < \pi \tag{118}$$

where ε is a quantity which may be diminished indefinitely.

One of the difficulties in writing

$$y = \sin x \sin\alpha + \sin 2x \sin 2\alpha + \cdots\cdots \tag{119}$$

in the Theory of Functions of a Real Variable is that this series has no "sum" when $n \to \infty$. It therefore becomes necessary to investigate its H-sum. For the sake of simplicity the H-sum of another, but similar, series will be discussed here, *i.e.*,

$$y_n = \frac{\sin(2n+1)\beta}{\sin\beta} = \frac{1}{2} + \cos 2\beta + \cos 4\beta + \cdots\cdots \tag{120}$$

since this series is considered in detail by Byerly. From this series it is simple to extend the discussion to the more general series (106), as (106) may be expressed as the sum of two series of the form (120). Byerly has plotted a curve showing the form of (120) when n is finite. It is made up of successive positive and negative arches of decreasing amplitude. As n is increased the number of arches between 0 and π increases. These arches are all bounded by the curves cosec β and $-$ cosec β, which remain constant as n is increased indefinitely. Thus we conceive of the H-sum of the series as the limit which is approached as the number of arches is increased indefinitely, which may be regarded as a curve filling the whole of the space between the boundaries cosec β and $-$ cosec β. Such a curve might be regarded as non-H-uniformly convergent at every point in the interval $0 < x < \pi$, and from a purely arithmetical point of view is meaningless. Looked at from the point of view of its integral between any two points in the interval, it has a definite meaning, which will be found useful later.

There is an interesting way of retaining the analogy with the present method of taking the sum of a series in the Theory of Functions of a Real Variable, and the taking of a "sum" to obtain these sums.

Let us assume, for the purpose of illustration, that we desire the "sum" of the series

$$y = \frac{\sin(2n+1)\beta}{\sin\beta} = \frac{1}{2} + \cos 2\beta + \cos 4\beta + \cdots\cdots \qquad (121)$$

when $x = 5°$. If this value of x is inserted in the series and the sum to n terms taken, we find,

$$\begin{aligned}
&S_0 = S_{17} = S_{36} = S_{53} = \cdots\cdots && = .500\\
&S_1 = S_{18} = S_{37} = S_{52} = \cdots\cdots && = 1.4848\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_7 = S_{10} = S_{43} = S_{46} = && = 5.5413\\
&S_8 = S_9 = S_{44} = S_{45} = && = 5.7149\\
&S_{13} = S_{33} = S_{54} = S_{71} = && = -.500\\
&S_{19} = S_{34} = S_{55} = S_{70} = && = -1.4848\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&S_{25} = S_{28} = S_{61} = S_{64} = && = -5.5413\\
&S_{26} = S_{27} = S_{62} = S_{63} = && = -5.7149
\end{aligned} \qquad (122)$$

Now we might call the numbers $\pm.5$, ± 1.4848, ± 2.4245, . . .,

± 5.7419 points of accumulation for this series and say that the sum of the series has these values when $x = 5°$. This would, however, be overlooking the fact that the H-sum of the series is the sum when x not only has the value $5°$, but also when it approaches $5°$. By taking points very close to $x = 5°$, we can obtain any number between cosec $5°$ and $-$ cosec $5°$ as a point of accumulation for the "sum" of the series. Hence the H-sum of the series to n terms as, $n \to \infty$, when x lies in the neighborhood of $5°$, has every point between cosec $5°$ and $-$ cosec $5°$ as a point of H-accumulation. It will similarly be found that for any other value of $x = x_1$, the H-sum has an infinite number of points of H-accumulation lying between cosec x_1 and $-$ cosec x_1. It is in this sense that the series can be regarded as an H-function of x. It must be remembered, however, that when it is considered thus as an H-function, it retains its other limiting properties as well as its characteristics of points of H-accumulation for each value of x.

The terms, point of accumulation, in the ordinary mathematical sense, and point of H-accumulation, used here are, as far as can be seen at present, identical. The latter is used, however, to avoid any unforeseen objections which might arise to the use of the former term. It may be remarked that the H-limit of the sum of a series, if we consider its arithmetical properties alone, is the aggregate of the aggregate of points of accumulation of the sum in the neighborhood of the various values of the independent variable in any given interval. In the Theory of Functions of a Real Variable, on the other hand, a limiting point is always a point of accumulation, whereas a point of accumulation need not necessarily be a limiting point.

It is not a very simple matter to give a definition of an H-function which will cover all the points noted above. The following definition may be regarded as too limited, but at present no alternative presents itself. As to the objections to the limited scope of the definition this may be said. When a definition deals with a limited class of things, these things may have very many properties in common, and thus numerous theorems can be derived relating to the things involved. As the definition is extended to deal with larger classes of things, the properties which they have in common are, in general, fewer, and thus a smaller number of the theorems which were derived for the

narrower class is true. Thus there seems to be the option of keeping a limited form of definition with many precise results, or enlarging the definition and obtaining a smaller number of what may be called more general results.

An H-function will now be defined as follows: Let x lie in the continuous interval $a \leq x \leq b$. Then if a formula or norm is given for determining the value of y in the neighborhood of every point x in this interval, y is said to be an H-function of x. In case that the value of y tends to more than one value in the neighborhood of any point x, a definite limiting process must be given to determine fully the characteristic properties of the H-function at this point. The term "in the neighborhood of" is to be understood to include the point itself as one point in its own neighborhood.

It will be noticed that the curve β above is defined by a norm and a limiting process in the latter part of the definition (101)

$$\beta = \underset{n \to \infty}{H\text{-limit}} \quad (\text{curve } B) \tag{123}$$

The H-function given by

$$y = \sin x \sin \alpha + \sin 2x \sin 2\alpha + \cdots \tag{124}$$

is defined by a formula.

This definition of an H-function obviously excludes such a definition as is given in the Theory of Functions of a Real Variable,

$$\left.\begin{array}{l} y = 1 \text{ when } x \text{ is rational} \\ y = 2 \text{ when } x \text{ is irrational} \end{array}\right\} \; 0 \leq x \leq 1 \tag{125}$$

since no limiting process is given to determine the characteristic properties of the function in the neighborhood of any point. In order to translate this into an H-function, it would be necessary to give a limiting process of which the following might be considered as an example. Divide the interval $0 \leq x \leq 1$ into n equal parts. In each of these parts take an irrational number according to a certain law. Set up the corresponding values of y at each of the rational and irrational points of x thus selected, and join each of the consecutive values of y by a straight line. The limit of this curve as $n \rightarrow \infty$ defines an H-function y_h which has properties in common with the function

$$\left.\begin{array}{l} y = 1 \text{ when } x \text{ is rational} \\ y = 2 \text{ when } x \text{ is irrational} \end{array}\right\} \; 0 \leq x \leq 1 \tag{126}$$

since the H-function has a pair of points identical with every pair of points given by (126). But the H-function has other properties in addition. For instance, it has values of y which do not appear in (126), and also, for example, $\int_0^1 y_h dx$ is equal to $3/2$ since this is true when n is finite.

It should be obvious, of course, that the consecutive points might also have been joined by other curves instead of straight lines, and these curves would still possess properties in common with the function as ordinarily defined by (126). If these curves are convex to the x-axis, $\int_0^1 y_h dx$ is greater than $3/2$, and if concave $\int_0^1 y_h dx$ is less than $3/2$. It is evident that these processes define entirely different H-functions and they must carefully be distinguished as such.

It has already been seen in connection with H-differentiation that

$$\left.\begin{array}{ll} y = (1-\xi)x & x \leq \xi \\ y = (1-x)\xi & x \geq \xi \end{array}\right\} \tag{127}$$

is permissible as the definition of an H-function. These two formulæ give the same value for y when $x = \xi$. The H-differential coefficient of (127) was found to be

$$\left.\begin{array}{lll} y = 1-\xi & & x < \xi \\ y = -\xi & & x > \xi \\ y = \dfrac{1}{1+p} - \xi & (0 \leq p \leq \infty) & x = \xi \end{array}\right\} \tag{128}$$

and this definition of an H-function does not present apparent difficulties. It was found, however, that the second, third and higher H-derivatives did present difficulties when definitions similar to those given above were used. The preferable way to define all these derivatives is now seen to be to give the H-function

$$\left.\begin{array}{ll} y = (1-\xi)x & x \leq \xi \\ y = (1-x)\xi & x \geq \xi \end{array}\right\} \tag{129}$$

and call them the first, second, third, etc., H-derivatives of this H-function. This makes their definition quite precise, and if the values to which these H-derivatives converge are required at any time, they will be found in the section dealing with H-differentiation.

DEFINITION OF AN H-LIMIT.

In the discussion on an H-function above it has been necessary to point out that the present conception of limit is based on only one of the properties of a function, namely, its numerical value. Of course, as it is defined in the Theory of Functions of a Real Variable, the only properties which a function has, in the last analysis, are numerical values which correspond to given numerical values of the independent variable, and the aggregate of these values, together with the one to one correspondence between them, go to make up the function. But, as far as actual work with the function goes, each value of the independent variable is completely isolated from any neighboring values. For instance, when it is desired to find the value of $y = f(x)$ at the point $x = x_1$, the value x_1 is inserted in $f(x)$, and the result obtained is the value of y corresponding to x_1. Similarly, in summing a series, the value of x at which the sum is desired is first inserted in the series, and the numerical series thus obtained summed as a series of pure numbers.

In the Theory of H-functions it has been repeatedly pointed out that in order to obtain the value of an H-function $F(x)$ "at the point $x = x_1$," it is not only necessary to insert the value x_1 in the H-function $F(x)$, but it is also essential to let x approach the point x_1 in an arbitrary manner, which is the same as considering the value of $F(x)$ when x lies in the neighborhood of x_1.

By connecting each point of x to all the others by means of these neighborhoods, the H-function takes on other properties in addition to the mere correspondence between numerical values. It might be said to take on a characteristic form. As an example of such a form, reference may again be made to the curve B above (Fig. 1), formed of the sides of the equilateral triangles on the n sections into which the interval $0 \leq x \leq 1$ is divided. As the number n is increased indefinitely, the length travelled along this curve as a point goes from $x = 0$ to $x = 1$ is 2, and this remains constant as n increased indefinitely. Thus we have

$$\lim_{n \to \infty} \begin{pmatrix} \text{Distance travelled along curve } B \text{ by} \\ \text{a point going from } x = 0 \text{ to } x = 1 \end{pmatrix} = 2 \qquad (130)$$

When the H-limit of the curve B is taken and the equation

$$H\text{-limit}_{n \to \infty} (\text{curve } B) = \beta \qquad (131)$$

is written, this latter H-limit equation includes (130).

As another example, consider the series

$$y = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\cdots\right] \tag{132}$$

When n is finite this curve oscillates around the line $y = 1$. As n is increased it oscillates a greater number of times around this line. When the H-limit is taken it continues to retain this property and thus

$$H\text{-limit}_{n \to \infty} [S_n(x)] = y \tag{133}$$

implies (among other things) that the curve y oscillates an infinite number of times around the line $y = 1$.

Thus the H-limit may be defined in general as the aggregate of the limits of all the characteristic properties of the H-function, where, for the present, these limits may be taken in the ordinary manner. It might appear to be not a very easy matter to define what "all the characteristic properties of an H-function" are. If, however, the suggestion offered in the next section on H-continuity is followed, and the operations adjusted so that we always have H-functions to deal with, this should not present any very great difficulty.

H-CONTINUITY.

Continuity does not seem to exist as a problem in the Theory of H-functions. It has already been pointed out how discontinuities may arise in the Theory of Functions due to the fact that the limiting process with regard to one variable is given precedence over the limiting process with respect to another variable. It has also been shown in several cases how such discontinuities may be avoided by adopting the rule that limiting processes with respect to each variable shall be taken simultaneously. The H-limit of any process has been defined as the aggregate of all the limits of the properties of the H-function under consideration. Thus if in attempting to define an H-function we do so by giving another set of H-functions whose H-limit is to be taken, and if these H-functions are continuous, then the H-function defined by the H-limiting process is by its very definition H-continuous.

It may be well to point out that the Heaviside unit function

$$\left.\begin{array}{rr} E1 = 0 & t < 0 \\ E1 = E & t > 0 \\ 0 \leq E1 \leq E & t = 0 \end{array}\right\} \tag{134}$$

which is H-continuous at $t = 0$ is not continuous according to the test for continuity used in the Theory of Functions, *i.e.*, $f(+0)$ is not equal to $f(-0)$.

There are some functions which perhaps might yet be regarded as really discontinuous. Consider $y = x \log x$. As x decreases from 1 to 0, y remains real. When, however, x becomes less than 0, y ceases to exist as a real quantity. To avoid having to regard such an H-function as discontinuous at $x = 0$, it seems best to regard y as having imaginary values when x is less than zero, and these imaginary values form a continuous sequence with the real values of y when x is greater than zero.

The question of H-continuity may perhaps be looked at from a slightly different viewpoint as follows. Let us exclude such definitions of H-function as

$$\left.\begin{array}{l} y = 0 \text{ when } x = \text{a rational number} \\ y = 1 \text{ when } x = \text{an irrational number} \end{array}\right\} \qquad (135)$$

which may be regarded as a deliberately defined discontinuous function. The objections to (135) as a definition of an H-function have already been pointed out above. Assume that we start with continuous functions or H-functions and apply to them various processes which we may call H-operations. If as the result of any such H-operation we arrive at a discontinuous function, *i.e.*, a function which can no longer be regarded as an H-function, it would seem proper that we should carefully examine such a process and see if the discontinuity did not arise from an actual discontinuity in the definition of the H-operation itself. For instance, in taking the sum of a series, the limiting process with respect to the variable x takes precedence over the limiting process with respect to n. Re-defining the sum process as an H-sum process in which neither of the limiting processes takes precedence over the other, may be regarded as the same as replacing the discontinuous definition of "sum" by the continuous definition of "H-sum." It is difficult to see how, if such a rule is adopted in framing or extending definitions, discontinuous H-functions can ever arise.

APPLICATION OF THE THEORY OF H-FUNCTIONS.

We will now endeavor to show how the Theory of H-functions as already developed may be used to develop the Heaviside Expansion Theorem. Assume that the operator p

denotes D/Dt, $p^2 = D^2/Dt^2$ and in general $p^n = D^n/Dt^n$. Then since the definition of H-integration is that it is the inverse process of H-differentiation, we have that if

$$py = Q \tag{136}$$

$$y = \frac{1}{p} Q \tag{137}$$

if we use the laws of algebra on p. But the solution of (136) is

$$y = \int Q\,dt + C \tag{138}$$

Hence if the arbitrary constant $C = 0$, then the symbol $1/p$ denotes the process of one H-integration applied to the H-function. Similarly $1/p^n$ denotes the application of n H-integrations to the H-function.

Let us now examine some very simple examples to outline in full the procedure. Consider the equation

$$(p + \alpha)\, y = E1 \tag{139}$$

where the function $E1$ is given by

$$\left.\begin{array}{rl} E1 = 0 & t < 0 \\ E1 = E & t > 0 \\ 0 \leq E1 \leq E & t = 0 \end{array}\right\} \tag{140}$$

We will now define the solution of (139) and (140) to be

$$y = \frac{E1}{\alpha} (1 - e^{-\alpha t}) \tag{141}$$

where $e^{-\alpha t}$ is to be considered merely as a symbol whose properties will now be considered, and which is assumed to have the property

$$(e^{-\alpha t})_{t=0} = 1. \tag{142}$$

It might possibly be better to use some other symbol here instead of $e^{-\alpha t}$ in order to differentiate it from the $e^{-\alpha t}$ used in the theory of functions. It is not thought, however, that this will occasion any confusion. This symbol will be used hereafter to denote the H-function unless specific mention is made to the contrary.

If (141) is inserted in (139) we get

$$\frac{pE1}{\alpha} (1 - e^{-\alpha t}) + E1\,(1 - e^{-\alpha t}) = E1. \tag{143}$$

Let

$$y_1 = \frac{pE1}{\alpha} (1 - e^{-\alpha t}) \tag{144}$$

and

$$y_2 = \frac{E1}{\alpha}(1 - e^{-\alpha t}) \tag{145}$$

Then y_2 is in its final form. Due to the fact that $E1 = 0$ when $t < 0$, y_2 is equal to zero when $t < 0$. When $t = 0$, $E1$ may have any value from 0 to E, but the value of $1 - e^{-\alpha t}$ is zero when $t = 0$ from (142), hence $y_2 = 0$, when $t = 0$. When $t > 0$, $E1 = E$ and the value of $y_2 = \frac{E}{\alpha}(1 - e^{-\alpha t})$ when $t > 0$. Thus y_2 can be represented graphically as in Fig. 3.

But from (144) $y_1 = py_2$ so that it is necessary to H-differentiate the H-function shown graphically in Fig. 3. This is easily

FIG. 3.

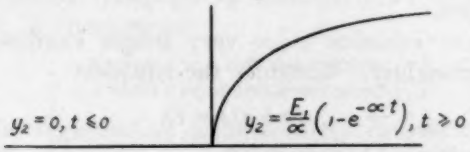
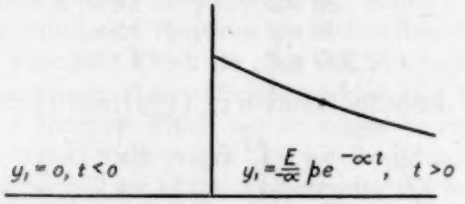


FIG. 4.



done in accordance with the rules already laid down, and the result is that y_1 may be represented graphically as shown in Fig. 4, where

$$\left.\begin{aligned} y_1 &= 0 && t < 0 \\ y_1 &= -\frac{E}{\alpha} pe^{-\alpha t}, && t > 0 \\ 0 \leq y_1 &\leq -\frac{E}{\alpha} pe^{-\alpha t} && t = 0 \end{aligned}\right\} \tag{146}$$

or using the notation of (140)

$$y_1 = -\frac{E1}{\alpha} pe^{-\alpha t} \tag{147}$$

In general p is to be assumed to operate only on what comes after it. Substituting this value of y_1 in (143), we get

$$-\frac{E1}{\alpha} pe^{-\alpha t} + E1 - E1e^{-\alpha t} = E1. \tag{148}$$

Hence

$$E1pe^{-\alpha t} = -\alpha E1e^{-\alpha t} \tag{149}$$

and on cancelling out the factor $E1$ common to both sides of the equation

$$pe^{-\alpha t} = -\alpha e^{-\alpha t}. \tag{150}$$

Thus we may replace (147) by

$$y_1 = E1e^{-\alpha t} = \frac{pE1}{\alpha}(1 - e^{-\alpha t}). \tag{151}$$

We might, if we had desired, have started initially with the definition of $e^{-\alpha t}$ by (150) and (142) and have proved from this that (141) is the solution of (139). It is a little easier and clearer to proceed along the lines given above.

Now consider the equation

$$(p + \alpha)y = (p + \beta)E1 \tag{152}$$

By reversing the order of operations which it is permissible to do in the Theory of H-functions

$$y = (p + \beta)\frac{E1}{\alpha}(1 - e^{-\alpha t}) \tag{153}$$

We have already shown in (151)

$$\frac{pE1}{\alpha}(1 - e^{-\alpha t}) = E1e^{-\alpha t} \tag{154}$$

Hence

$$y = E1e^{-\alpha t} + \frac{\beta E1}{\alpha}(1 - e^{-\alpha t}) \tag{155}$$

or

$$y = \left[\frac{\beta}{\alpha} + \frac{(\alpha - \beta)}{\alpha}e^{-\alpha t}\right]E1 \tag{156}$$

which is the solution obtained by the Heaviside expansion theorem.

As a further example consider the equation

$$(p + \alpha)y = (p + \beta)(p + \gamma)E1 \tag{157}$$

Then, as before, the solution is by definition

$$y = \frac{(p + \beta)(p + \gamma)E1}{\alpha}(1 - e^{-\alpha t}) \tag{158}$$

and by operating with $(p + \beta)$ first, we get from (155)

$$y = (p + \gamma)\left[E1e^{-\alpha t} + \frac{\beta E1}{\alpha}(1 - e^{-\alpha t})\right] \tag{159}$$

which by operating on the second term in the usual manner gives

$$y = (p+\gamma)\,E1e^{-\alpha t} + \beta E1e^{-\alpha t} + \frac{\beta\gamma}{\alpha}\,E1\,(1-e^{-\alpha t}). \tag{160}$$

The first term becomes

$$pE1e^{-\alpha t} + \gamma E1e^{-\alpha t} = y_1 + y_2 \tag{161}$$

where

$$y_1 = pE1e^{-\alpha t} \tag{162}$$

Now the term $y_1 = pE1e^{-\alpha t}$ denotes the H-differential coefficient of the function already shown graphically in Fig. 4. This has already been discussed in connection with H-differentiation. Its value is zero when $t < 0$, $-\alpha Ee^{-\alpha t}$ when $t > 0$ and $0 \leq pE1e^{-\alpha t} \leq E1e^{-\alpha t}\infty^1$ when $t = 0$ where the symbol $E1e^{-\alpha t}\infty^{-1}$ has the meaning already assigned to it in the section on H-integration.

Thus, in general, it can be shown that the solution of (157), where p occurs in a higher power on the right-hand side than on the left-hand side of the equation, contains what Heaviside calls an impulsive function, *i.e.*, the value of y at $t = 0$ varies from 0 to ∞ when the degree of p on the right-hand side of the equation is greater by unity than the degree of p on the left, and varies from $-\infty$ to $+\infty$ when the degree of p on the right-hand side exceeds the degree of p on the left-hand side by more than unity.

It may be desirable at this point to call attention to the main differences between the method adopted here and that of the Theory of Functions of a Real Variable.

The H-function is *defined* by (141) and (142) to be the solution of (139). It is not defined by an infinite series. In (150) it has been shown that the H-function so defined has the property

$$pe^{-\alpha t} = -\alpha e^{-\alpha t}. \tag{150}$$

By adopting such a definition we avoid having to use in our work any quantity ε, which though it may become as small as we please can never become zero. Thus in all our equations we can use the sign of equality and not the sign $\longleftrightarrow$ which means "converges to."

It may be argued, however, that nothing has been gained by this since the symbol $e^{-\alpha t}$ when thus defined means no more than the original equation.

This is perfectly true, but we are no better off if we define

$\epsilon^{-\alpha t}$ by means of an infinite series. For instance, suppose $e^{-\alpha t}$ to be given by the series

$$e^{-\alpha t} = 1 - \alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \cdots\cdots \tag{163}$$

then all that we can prove is that in the region of any particular value of t say t_1, which we choose to name, the value of the sum of this series differs by a quantity ε, which may be named as small as we please, from the value of y obtained from

$$\frac{dy}{dt} + \alpha y = E \tag{164}$$

under the terminal condition

$$y = \frac{E}{\alpha} \text{ when } t = 0 \tag{165}$$

or we may write

$$y \longleftrightarrow E e^{-\alpha t} \tag{166}$$

where $e^{-\alpha t}$ is given by (163).

If on the contrary $Ee^{-\alpha t}$ is defined to be the solution of (164) and (165), then

$$y = Ee^{-\alpha t} \tag{167}$$

and

$$e^{-\alpha t} \longleftrightarrow 1 - \alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \cdots \tag{168}$$

Now we find in the Theory of H-functions that by using (167) and (168), it is possible to get very many more exact relations than by using (163) and (166). For instance, let us consider a linear equation of the second degree with constant coefficients written in the form

$$(p + \alpha)(p + \beta) y = E1 \tag{169}$$

Then as shown in the paper in this JOURNAL, June, 1923, this equation can be solved by solving two equations of the first degree, and then applying the algebraic process of elimination. By going through this process we get

$$y = E\left[\frac{1}{\alpha\beta} + \frac{e^{-\alpha t}}{\alpha(\alpha - \beta)} + \frac{e^{-\beta t}}{\beta(\beta - \alpha)}\right] \tag{170}$$

as shown there. Thus we can write the sign of equality here instead of the sign $\longleftrightarrow$ which would have been necessary if $e^{-\alpha t}$ had been defined by an infinite series.

Let us now attempt to solve the equation

$$(p+\alpha)(p+\beta)y = E1 \tag{171}$$

along the same lines as those used by Heaviside in his E.M.T.

We have already defined in (140) what $E1$ is, and also in (141) and (139) that the solution of the equation

$$(p+\alpha)y = E1 \tag{139}$$

is

$$y = \frac{E1}{\alpha}(1 - e^{-\alpha t}). \tag{141}$$

We may also write (139) as Heaviside writes it

$$y = \frac{E1}{p+\alpha} \tag{172}$$

Hence equating (141) and (172)

$$\frac{E1}{p+\alpha} = \frac{E1}{\alpha}(1 - e^{-\alpha t}). \tag{173}$$

To solve (171) we may write it

$$y = \frac{E1}{(p+\alpha)(p+\beta)} = \frac{E1[p+\beta-p-\alpha]}{(\beta-\alpha)(p+\alpha)(p+\beta)} \tag{174}$$

Hence

$$y = \frac{E1}{(\beta-\alpha)(p+\alpha)} - \frac{E1}{(\beta-\alpha)(p+\beta)} \tag{175}$$

giving from (173)

$$y = \frac{E1}{(\beta-\alpha)\alpha}(1 - e^{-\alpha t}) - \frac{E1}{(\beta-\alpha)\beta}(1 - e^{-\beta t}) \tag{176}$$

which can be simplified to give

$$y = \left[\frac{1}{\alpha\beta} + \frac{e^{-\alpha t}}{\alpha(\alpha-\beta)} + \frac{e^{-\beta t}}{\beta(\beta-\alpha)}\right] E1 \tag{177}$$

which is the direct result of the application of the Heaviside expansion theorem.

It may be objected that the operation of cancelling out the common terms from the numerator and denominator in the transition from (174) to (175) is not permissible. It can be shown to be permissible as follows. Consider the equation

$$(p+\alpha)y = (p+\alpha)E1. \tag{178}$$

The solution of this, if we write it as Heaviside wrote it, is

$$y = \frac{p+\alpha}{p+\alpha} E1 \tag{179}$$

and if it is permissible to cancel out the common factor then

$$y = E1 \tag{180}$$

The solution of (178) by the *definitions* given above is

$$y = \frac{(p+\alpha)E1}{\alpha}(1-e^{-\alpha t})$$
$$= -\frac{E1}{\alpha}pe^{-\alpha t} + E1 - E1e^{-\alpha t}$$
$$= E1 \tag{181}$$

and in general this procedure may be used with any more complicated problem.

Let us now attempt to generalize the solution of (171) to the case where there is an operator acting on $E1$ on the right-hand

FIG. 5.

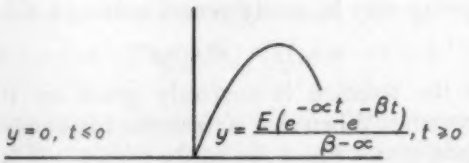
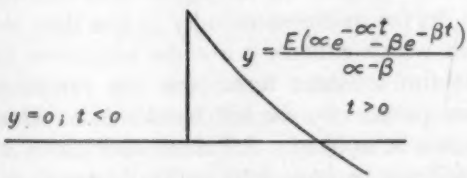


FIG. 6.



side of the equation. We shall only consider the highest power of the operator in the result. Let it be p^n. If $n = 1$, we have the same case as before, and (176) becomes

$$y = -\frac{E1}{\alpha-\beta}e^{-\alpha t} - \frac{E1}{\beta-\alpha}e^{-\beta t}$$
$$= \frac{E1(e^{-\alpha t} - e^{-\beta t})}{\beta-\alpha} \tag{182}$$

If $n = 2$, we have to evaluate

$$y = \frac{pE1(e^{-\alpha t} - e^{-\beta t})}{\beta-\alpha} \tag{183}$$

The curve to be H-differentiated will be as shown in Fig. 5, and by taking the H-derivative, Fig. 6 is obtained, which is the H-function given by

$$\left.\begin{aligned} y &= 0 && t<0 \\ y &= \frac{E(\alpha e^{-\alpha t}-\beta e^{-\beta t})}{\alpha-\beta} && t>0 \\ \text{and}\quad 0 \leq y &\leq \frac{E(\alpha e^{-\alpha t}-\beta e^{-\beta t})}{\alpha-\beta} && t=0 \end{aligned}\right\} \tag{184}$$

and the three expressions in (184) may be combined into one in the notation already used by writing

$$y = \frac{E1(\alpha e^{-\alpha t}-\beta e^{-\beta t})}{\alpha-\beta} \tag{185}$$

If $n=3$ we have to take the H-derivative of this H-function, and it is readily found that the H-derivative is impulsive, *i.e.*, when $t=0$, y ranges from 0 to ∞, but when $t \neq 0$ it is finite.

The foregoing may be easily generalized to solve the equation

$$(p^m + a_1 p^{m-1} + \cdots\cdots + a_m)y = (p^n + b_1 p^{n-1} + \cdots\cdots + b_n)E1. \tag{186}$$

When $m \geq n$ the solution is correctly given by the Heaviside expansion theorem. When $m < n$ there are impulsive H-functions occurring in addition. These are easily found if it is remembered that the expansion in partial fractions only holds when $m \geq n$. If $m < n$ divide the operator on the right-hand side by that on the left, until the degree of the remainder is less than m. The terms in the quotient which contain p are the impulsive functions, and there remains the constant term plus the remainder, which is divided by the operator on the left-hand side to which the expansion theorem can be applied. All these operations are easily performed and will not be gone into in detail here.

A word may be said to point out the difference in procedure here and in Heaviside's own work. Heaviside wrote the equation

$$(p+\alpha)y = E1 \tag{187}$$

in the form

$$y = \frac{E1}{p+\alpha} \tag{188}$$

and said "here is the solution." This is perfectly correct. Then he proceeded to interpret the result by a process he called "algebrizing," *i.e.*, he expanded the fraction in powers of $1/p$ and got

$$y = \left(\frac{1}{p} - \frac{\alpha}{p^2} + \frac{\alpha^2}{p^3} - \cdots\cdots\right)E1 \tag{189}$$

and by performing the integrations he got

$$y = \left(t - \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^3}{3!} - \frac{\alpha^3 t^4}{4!} + \cdots\right)E1 \tag{190}$$

or

$$y = \frac{E1}{\alpha}(1 - e^{-\alpha t}) \tag{191}$$

Now in the Theory of H-functions (188) does not mean the same thing as (189). All that can be shown is that the difference between the value of y as calculated from (189) and the value of y as found from (188) is less than any quantity ε we choose to name.

If Heaviside had put his argument a little differently by saying, "The solution of (187) is

$$y = \frac{E1}{p+\alpha} \tag{192}$$

or if you wish to write it another way,

$$y = \frac{E1}{\alpha}(1 - e^{-\alpha t}) \tag{193}$$

(192) containing the symbol $E1/(p+\alpha)$, and (193) containing the symbol $e^{-\alpha t}$, neither of which symbols we understand fully. However, if somebody wants to get numerical results, I will 'algebrize' these expressions for him, and give him the series

$$e^{-\alpha t} \longleftrightarrow 1 - \alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \cdots \tag{194}$$

from which he can get numerical values with an error smaller than any quantity ε he chooses to name, by taking a sufficient number of terms," then I do not think any objection could be made to his expansion theorem. This is the attitude taken here.

DIRECT APPLICATION OF THE HEAVISIDE UNIT FUNCTION TO THE SOLUTION OF A DIFFERENTIAL EQUATION.

While the geometrical methods of establishing these results may be interesting in showing the mechanism involved, in actual work the development may be greatly shortened by considering $e^{-\alpha t}1$ as the product of the two H-functions $e^{-\alpha t}$ and 1. From the formula for differentiation of a product

$$pe^{-\alpha t}1 = 1pe^{-\alpha t} + e^{-\alpha t}p1 \tag{195}$$

and since it has already been shown or defined that

$$pe^{-\alpha t} = -\alpha e^{-\alpha t} \tag{196}$$

it follows that

$$pe^{-\alpha t}1 = -\alpha e^{-\alpha t}1 + e^{-\alpha t}p1 \tag{197}$$

but $e^{-\alpha t}\,p1$ is zero everywhere except at $t=0$, where its value is $p1$ multiplied by the value of $e^{-\alpha t}$ when $t=0$, which is unity. Hence

$$pe^{-\alpha t}1 = -\alpha e^{-\alpha t}1 + p1 \tag{198}$$

As an example, let us show that the solution of

$$\frac{Dy}{Dt} + \alpha y = E1 \tag{199}$$

is

$$y = \frac{E1}{\alpha}(1 - e^{-\alpha t}). \tag{200}$$

We have

$$\begin{aligned}\frac{Dy}{Dt} &= \frac{D}{Dt}\left(\frac{E}{\alpha}1 - \frac{E}{\alpha}e^{-\alpha t}1\right)\\ &= \frac{E}{\alpha}p1 - \frac{E}{\alpha}[-\alpha e^{-\alpha t}1 + p1]\\ &= Ee^{-\alpha t}1\end{aligned} \tag{201}$$

Hence

$$\begin{aligned}\frac{Dy}{Dt} + \alpha y &= Ee^{-\alpha t}1 + E1 - Ee^{-\alpha t}1\\ &= E1\end{aligned}$$

thus showing that (200) is the solution of (199) when $E1$ is interpreted as in (140).

Now consider the same circuit, but let the E.M.F. be short-circuited at the instant $t=0$. Then we have to show that the solution of

$$(p+\alpha)y = E - E1 \tag{202}$$

which is usually written in the Theory of Functions

$$(p+\alpha)y = 0 \tag{203}$$

is

$$y = \frac{E}{\alpha}e^{-\alpha t}1 + \frac{E}{\alpha}(1-1) \tag{204}$$

From (204)

$$\begin{aligned}py &= -Ee^{-\alpha t}1 + \frac{E}{\alpha}p1 - \frac{E}{\alpha}p1\\ &= -Ee^{-\alpha t}1\end{aligned} \tag{205}$$

(Note: $p1$ is written equal to zero since 1 is a constant for all time and is not the Heaviside unit function 1). Hence

$$(p+\alpha)y = -Ee^{-\alpha t}1 + Ee^{-\alpha t}1 + E(1-1) = E(1-1) \tag{206}$$

The part played by the H-function 1 in this method of solu-

tion may perhaps be seen more clearly as follows. Consider the equation

$$\frac{dy}{dt} + \alpha y = E1 \tag{206a}$$

where $E1$ is given by (140). The general solution of (206a) is

$$\left.\begin{aligned} i &= i_1 e^{-\alpha(t-t_1)} + \frac{E}{\alpha}(1 - e^{-\alpha t}) \quad (t > 0) \\ i &= i_1 e^{-\alpha(t-t_1)} \qquad\qquad\qquad\quad (t < 0) \end{aligned}\right\} \tag{206b}$$

or as it may be written

$$i = i_1 e^{-\alpha(t-t_1)} + \frac{E1}{\alpha}(1 - e^{-\alpha t}) \quad \left(t \gtreqless 0\right) \tag{206c}$$

where a current i_1 is assumed to flow in the circuit at any time $t_1 < 0$ (it is immaterial how this current got there). If t_1 is a very large negative number the value of $e^{-\alpha(t-t_1)}$ approaches zero, provided that i_1 approaches a finite constant value as $t_1 \to -\infty$. In such a case, whatever disturbance may have been present at a very large negative value of time, will tend to be damped out at the time $t = 0$. Looking at the solution from this point of view we may say that it is the part of the specification of $E1$ in (140) which says $E1 = 0$ when $t < 0$, which automatically ensures $i = 0$ when $t = 0$, and thus there is no necessity for this latter statement since it is implicitly contained in the other data. In an undamped circuit, however, such as one containing inductance and capacity only (an ideal case), there is thus no dissipation of energy, it would appear that the additional specification must be given that as $t_1 \to -\infty$ the circuit was in a state of rest.

Another method of considering the solution would perhaps be from the point of view of cause and effect. Since the voltage $E1$ is zero when $t < 0$, it cannot produce any effects when $t < 0$. Thus in (206c) the second term may be considered as marked with the symbol 1 to denote the part of the solution due to the voltage $E1$ and the first part which is not multiplied by 1 is due to other causes and hence need not be considered if we are only interested in the effects produced by $E1$.

APPLICATION TO CONTINUOUS SYSTEMS.

We will now consider the application of these principles to partial differential equations, as it is in this region that the Heaviside operational method is particularly useful. The application will be shown by means of specific examples and from these

the general method should be obvious. The first three examples are solved, using forms of equations similar to those used by Heaviside in the solution of such problems. These examples are all one-dimensional ones, that is, they involve only one space coördinate in addition to the time coördinate. The first two examples are chosen to illustrate a suggested classification of Green's functions. The third example is given to show how the expansion theorem works even when the properties of the functions involved (Bessel functions in this case) are not as familiar as the trigonometric functions.

By means of a generalization of procedure, of which the actual method used by Heaviside is a particular case, the solution for the Green's function is obtained not only in two- and three-dimensional heat problems, but also in potential problems as well. The application of well-known theorems by means of which it is possible to derive, from the Green's function, solutions under other arbitrary conditions is illustrated by two particular cases.

(To be continued.)

Turpentine and Rosin. (Clip Sheet, U. S. Department of Agriculture, No. 381, October 12, 1925.)—The turpentine and rosin industry is of considerable importance in the United States, as indicated by figures recently put out by the Bureau of Chemistry. The various industries which use these products in their manufacturing reported that during the calendar year 1924 they used a total of 6,739,000 gallons of turpentine and 864,850 round barrels of rosin, The manufacturers included in these reports are makers of paint, varnish, soap, paper, rosin oil, pitch, printing ink, shoe polish, leather dressings, sealing wax, fly paper, insulating material, oils, greases, linoleum, roofing, automobiles and wagons, malleable iron and steel, pharmaceuticals, woodenware, and the operators of shipyards. Paint and varnish, shoe polish, leather dressings, and the automobile industry use the greater part of the turpentine, while the paper and paper size, varnish, soap, rosin oil, pitch, and printing-ink makers use by far the larger part of the rosin.

Wave-lengths of Additional Lines in the Many-lined Spectrum of Hydrogen. T. TANAKA. (*Proc. Roy. Soc.*, A 748.)—No less than 565 weak lines, not set down in Merton and Barratt's tables, were found on two plates taken by Professor Merton from a vacuum tube giving the hydrogen spectrum. Ten or more lines attributed to hydrogen by other experimenters were not found on these plates. The range in wave-lengths is from 6572.01 Ångström units to 3295.48, in air in both cases. G. F. S.

NOTES FROM THE RESEARCH LABORATORY, EASTMAN KODAK COMPANY.*

THE LAWS OF DEVELOPMENT OF X-RAY FILMS.[1]

By R. B. Wilsey.

THE characteristic curve of " Duplitized " X-ray film does not show a straight line portion in the range of useful densities, the average slope of the curve over the density range 0.5 – 2.0 is taken as a measure of the contrast, and speed is proportional to the reciprocal of the exposure required to give a density of one, with standard development.

Contrast is found to increase with development time to a maximum and then decreases owing to the growth of fog over the lower densities. The best development time is that giving the maximum contrast, and there is a range of development times on either side of this value for which the contrast does not differ greatly from the maximum. As a rule, the lower the fog, the greater is the contrast and the longer is the range of satisfactory development times. The relation between exposure and development time to give equal density of radiograph is shown graphically.

The correct time of development with a partially exhausted X-ray developer can be determined by measuring the time of appearance and multiplying by the appropriate Watkin's factor.

X-ray films can be developed at temperatures up to 90° F. by the use of hardening bath between development and fixing. Detailed directions for this process are given.

PLASTICITY IN RELATION TO CELLULOSE AND CELLULOSE DERIVATIVES.[2]

By S. E. Sheppard and E. K. Carver.

MOST of the earlier workers on the viscosity of solutions of cellulose and cellulose derivatives have neglected the effect of velocity of shear on apparent viscosity. This effect is fairly large

* Communicated by the Director.

[1] Communication No. 239 from the Research Laboratories of the Eastman Kodak Company and published in *Radiology*, **5**: 237, 1925.

[2] Communication No. 225 from the Research Laboratories of the Eastman Kodak Company and published in *J. Phys. Chem.*, **29**: 1244, 1925.

with these substances, showing that they are true plastics. Plasticity curves for solutions of nitrocellulose in cuprammonium hydroxide and for nitrocellulose in various solvents show that the apparent viscosity may vary many fold according to the rate of shear used.

The measurements by Peirce of the moduli of rigidity of single cotton hairs ($0.23 \times 10''$ dynes/cm.^{2}), compared with those of the authors on cellulose nitrate and acetate films ($0.07 \times 10''$ dynes/cm.^{2}), seem to indicate that the rigidity is decreased by esterification, but in view of uncontrolled humidity conditions the measurements are not exactly comparable.

Stress-strain diagrams for cellulose ester films show an initial elastic portion, a region of plastic flow, and sometimes a secondary elastic portion. A film that has been repeatedly stretched does not show the same region of plastic flow as an unstretched film. Under steady loads, nitrocellulose films flow with a velocity increasing as a high power of the load. The behavior is like that of the ductile metals rather than that of rubber, although rubber, an organogel, might be supposed to resemble cellulose ester gels, more than the metals do. The mechanical behavior, then, must depend on the ultimate particle rather than on the structure. The dynamical model of Shorter and the theories of Mardles to explain the elastic behavior of fibres and gels are steps in the right direction.

AN AUTOMATIC RECORDER FOR MEASURING SIZE-FREQUENCY DISTRIBUTION OF GRAINS.[3]

By R. H. Lambert and E. P. Wightman.

THE paper describes a photographic recorder for measuring the motion of the meniscus of a column of liquid in the side arm of a *U*-type sedimentation tube. Light will pass through the tube filled with the liquid and focus on a strip of sensitive paper under it, but an empty portion of the tube is nearly opaque to the light at the position of the paper. Ordinary sensitized paper may be used, which can be obtained of such a length and width as will be suitable for a desired experiment. An experiment may proceed over an almost indefinitely extended time. A control device for making the exposure is described. The mathematical method

[3] Communication No. 244 from the Research Laboratories of the Eastman Kodak Company and published in the *J. Opt. Soc. Am.*, 11 : 393, 1925.

for correcting data is given. An actual sedimentation record is also presented.

ON THE RELATION BETWEEN TIME AND INTENSITY IN PHOTOGRAPHIC EXPOSURE. II.[4]

By L. A. Jones and E. Huse.

PREVIOUSLY published results on this subject showed rather conclusively that, (*a*) there was a definite failure of the reciprocity law, (*b*) the Schwarzschild equation $E = It^p$, would not fit the observed values, and (*c*) the experimental determination of the value of "optimal" intensity was subject to great uncertainty due to the flatness of a large portion of the D-log I curves.

Further work has been completed on three different materials and the results obtained bear out substantially those previously reported. The materials chosen for this second series of experiments were Seed 30 and Seed Graflex plates, and Eastman motion picture negative film.

An attempt was made, in the case of the results obtained with these materials, to interpret the failure of the reciprocity law in terms of the Kron equation, *i.e.*,

$$N = I \cdot t \cdot 10^{-a}\,[(\log I/I_0)^2 + 1]^{\frac{1}{2}}.$$

This formula when applied to some of these experimental data gives very satisfactory agreement. In the case of one of the materials investigated, Kron's formula could not be applied with the same certainty as with the others, since in making the exposures, sufficiently high values of intensity to make possible the definite determination of log I_0 were not used.

INCANDESCENT TUNGSTEN LAMP INSTALLATION FOR ILLUMINATING COLOR MOTION PICTURE STUDIO.[5]

By L. A. Jones.

THE advantages and disadvantages of various types of artificial illuminants for use in studios devoted to the production of color motion pictures are discussed briefly in the introductory part of the paper. The remainder of the paper consists of a description

[4] Communication No. 233 from the Research Laboratories of the Eastman Kodak Company and published in *J. Opt. Soc. Am.*, 11 : 319, 1925.

[5] Communication No. 238 from the Research Laboratories of the Eastman Kodak Company and published in *Trans. Soc. Mot. Pict. Eng.*, No. 22 : 25, 1925.

of the tungsten lamp equipment installed in the motion picture studio at the Eastman Theatre and School of Music. The design and performance of the reflector is treated in detail. This reflector consists of sixteen trapezoidal-shaped, silvered glass mirrors mounted in an aluminum shell. Three-thousand-watt tungsten lamps are used. Each lamp is individually cooled by means of a small blower mounted directly upon the cap of the reflector shell. This blower delivers a current of air which flows over the base of the lamp and sweeps out from the interior of the reflector the accumulation of heated air. Three of these sources, each equipped with an individual reflector, are mounted on an angle-iron frame and manipulated as a unit. An overhead gridiron structure supports the hoists and carriages for the manipulation of these units. Fifteen such units (carrying forty-five 3000-watt lamps) are provided as an overhead equipment. Three 75-ampere, high-intensity arcs are used for spotting. These are mounted as portable floor unit and carriage and driven from suitably situated wall pockets. The electrical distribution system and switch-board are discussed in detail. Some illumination measurements showing the levels obtainable are given. The paper is illustrated with diagrams and photographs showing the various elements and the complete installation.

Glycerin as a Radiator Compound.—Professor Keyes, of the Massachusetts Institute of Technology, who has had a large experience both in operating his own car and in supervising the operation of a fleet of trucks belonging to a company of which he is an officer, expresses entire confidence in mixtures of glycerin and water as anti-freeze liquids. Professor Maxwell, of the University of Alabama, also holds this opinion. Commercial glycerin may, however, be contaminated with fatty acids and therefore corrosive. A distilled article is now in the market, which is free from corrosive impurities, and will be furnished ready for use, the proportions of water and glycerin being adapted to the special use. The particular advantage that glycerin possesses over alcohol is that there is no appreciable loss by evaporation. In addition, the present legal restrictions on the sale of alcohol even when denatured are entirely avoided. The glycerin liquid is more expensive than the other solutions now in use, but only one filling is needed for the whole winter season, thus making the cost per season less than that of other methods. The solution will be marketed in containers. It has a boiling-point not markedly above that of water, and on the other hand a freezing-point much lower. H. L.

NOTES FROM THE U. S. BUREAU OF CHEMISTRY.*

HYDROGEN-ION CONCENTRATION IN BEET SUGAR FACTORY JUICES.[1]

A REPORT OF A PRELIMINARY STUDY OF pH VALUES IN SUGAR FACTORY LIQUORS, WITH SOME CONCLUSIONS.

By H. S. Paine, B. C. Sibley, and J. C. Keane.

[ABSTRACT.]

SYSTEMATIC electrometric measurements of the pH of diffusion, first and second carbonation, and third saturation juices were made in two beet sugar factories during the 1923 campaign and were compared with the titration values of the same samples. When the data are chronologically represented on coördinate paper, it is seen that, while the general contours of the corresponding pH and titration curves are somewhat similar, these two sets of values do not vary in a constantly similar manner. Determination of pH reveals significant differences which are not apparent from titration data.

Determination of pH in beet sugar factories has the following possible advantages: (1) Closer control of carbonation and sulphitation is possible. (2) Control by pH determination may be more rapid and less laborious than control by titration, provided that either a sufficiently simple colorimetric method or an automatic recording electrometric method is found to be satisfactory for the determination of pH. (3) It may be possible to apply pH control constructively so as to increase average non-sugar elimination. (4) pH measurements may be of value in connection with the analysis of ash for investigating certain factory problems involving salts. It must be understood, however, that further investigation is required in order to definitely determine whether or not these possible advantages can be realized.

Thymol blue and thymolphthalein test papers appear to be suitable in some cases but not in others. The general applicability of these test papers remains to be determined upon the basis of more extensive trial.

* Communicated by the Chief of the Bureau.

[1] Published in *Facts about Sugar*, **20**, No. 33 (Aug. 15, 1925).

PROTEINS OF THE COTTONSEED.[2]

By D. Breese Jones and Frank A. Csonka.

[ABSTRACT.]

BY extracting finely ground cottonseed kernels (hull-free) with benzene, nearly all of the fatty and resinous substances and much of the coloring material are eliminated, thus facilitating the subsequent extraction of the proteins by different solvents. For this purpose benzene is far superior to ether.

Two globulins, designated as α- and β-globulins, have been separated from a sodium chloride extract of cottonseed in yields of 2.59 and 16.00 per cent., respectively. The α-globulin was precipitated by the addition of ammonium sulphate to 0.4 to 0.5 saturation. It coagulated at 95–97° C. The β-globulin separated at a saturation of 0.7 to 0.8, but only when the extract had been previously diluted with water. It coagulated at 92–93° C.

Two preparations having similar properties and composition were separated from the sodium chloride extract by heating to 62° and 85° C. These fractions yielded 68.2 and 67.52 per cent. of ash, which had the following percentage composition: P_2O_5, 57.29; CaO, 9.71; MgO, 16.62; and Na_2O, 13.90.

A pentose protein was also isolated (2.08 per cent.), which contained 16.57 per cent. of pentose, 0.194 per cent. of phosphorus, and 12.64 per cent. of nitrogen.

The distribution of nitrogen in the proteins isolated was determined according to the Van Slyke method.

A small quantity of a substance having the properties of a glutelin was also isolated.

An attempt to isolate a nucleic acid gave negative results.

PROTEINS OF THE BARK OF THE COMMON LOCUST TREE, ROBINIA PSEUDACACIA.[3]

1. ENZYMES ASSOCIATED WITH THE PROTEINS: THE COMPOSITION, PROPERTIES, NITROGEN DISTRIBUTION, AND SOME OF THE AMINO ACIDS OF THE ALBUMIN.

By D. Breese Jones, C. E. F. Gersdorff, and O. Moeller.

THERE have been isolated from the air-dried inner bark of the common locust tree (*Robinia pseudacacia*) 2.52 per cent. of albumin and 1.38 per cent. of globulin (calculated on the basis of

[2] Published in *J. Biol. Chem.*, **64**, No. 3 (July, 1925), pp. 673–683.

[3] Published in *J. Biol. Chem.*, **64**, No. 3 (July, 1925), pp. 655–671.

moisture-free bark). A significant quantity of a substance having the properties of a proteose was also obtained. Successive exhaustive extractions with distilled water, 10 per cent. sodium chloride solution, 70 per cent. alcohol, and 0.5 per cent. sodium hydroxide removed from the coarsely ground bark 66.06 per cent. of the total nitrogen, equivalent as protein ($N \times 6.25$) to 11.56 per cent. of the bark. The small quantities of nitrogen extracted by alcohol and alkali were non-protein in character, showing the absence in the bark of proteins of the prolamin or glutelin type.

By making a salt extract of the bark 0.44 saturated by addition of ammonium sulphate, practically all of the protein was precipitated. The mixed proteins thus obtained had enzymic properties, capable of decomposing urea and amygdalin. Working with the isolated proteins, the enzymes were found to be associated with the globulin and not with the albumin or proteose.

The albumin, either in aqueous or salt solution, coagulated at 62–63° C. and was found to have the following percentage composition: C, 54.52; H, 6.83; N, 14.77; and S, 0.80. Analysis by the Van Slyke method showed it to have the following percentages of basic amino acids: Arginine, 4.39; histidine, 1.74; lysine, 5.45; and cystine, 1.37. Colorimetric estimations of cystine, tryosine, and tryptophane gave 1.03, 6.27, and 4.18 per cent., respectively. There was obtained by actual isolation 7.72 per cent. of aspartic acid and 4.48 per cent. of glutamic acid.

SUGAR-CANE SYRUP MANUFACTURE, Bulletin 1370, compiled by H. S. Paine and C. F. Walton, Jr.

The purpose of this bulletin, which will appear early in October, is to describe the manufacture of sugar-cane syrup in a comprehensive manner and in this way to assemble existing knowledge of the subject in one publication. It is largely a compilation of data from various sources, some of which have been published in technical bulletins and journals. Much new material, such as cost data, descriptions of model plants and apparatus, supplemented by sketches, and a comparison of the different methods of manufacture, is included.

The problems of production, from the cultivation of the cane to the marketing of the syrup, are described. Several methods for clarifying the juice are described in detail, these descriptions being

supplemented by cost data and a consideration of the relationship between the method of manufacture and the quality of the syrup.

The bulletin discusses the relationship between the cultural conditions under which sugar-cane is grown and the quality and yield of syrup. The factors which determine the most economical size of a new syrup plant are set forth in detail. The prevention of crystallization by the use of invertase, a process developed by the bureau, is fully described. Data on the composition, food value, and keeping quality of cane syrup are also given, as well as information regarding the production of uniform grades and the question of coöperative marketing.

Rat Extermination. (Clip Sheet, U. S. Department of Agriculture, No. 381, October 12, 1925.)—Rats living about the farmyard of a home near Eureka Lake, Kans., were recently given a taste of modern warfare by a representative of the Biological Survey, coöperating with the extension service of the agricultural college, whose time is given to rat-control work and the demonstration of the best means of eliminating this pest. Canned salmon, apples, and fresh eggs, seasoned with barium carbonate, were first rationed out along the known runways and haunts of the rats. Then came a gas attack for those that had survived. A small hand duster loaded with calcium-cyanide dust was used for gassing all the burrows and hiding places. Some of the rats plunged into the dust barrage only to die at the operator's feet. A week later the gas treatment was repeated, and the premises are now reported absolutely free of rats.

Chaulmoogra Oil.—TADAICHI HASHIMOTO, of Stanford University (*Jour. Am. Chem. Soc.*, 1925, **47**, 2325–2333), has studied the constituents of chaulmoogra oil which is used as a specific remedy in the treatment of leprosy. The oil, which is obtained from the seeds of certain trees, contains chaulmoogric acid, hydnocarpic acid, taraktogenic acid ($C_{26}H_{60}O_6$), isogadoleic acid ($C_{20}H_{38}O_2$), a lactone-like substance, a brown resinous material, two unidentified solids which probably are acids, and an acid which possibly is arachidic acid.

J. S. H.

Vitamin A Content of Fresh Eggs.—JOSEPH C. MURPHY and D. BREESE JONES (*Jour. Agric. Research*, 1924, **29**, 253–257) have studied the vitamin A content of fresh hen's eggs, using young albino rats as experimental animals. Sufficient vitamin A to produce normal growth was present in a daily dose of 0.50 to 0.75 gram of fresh whole egg. A smaller quantity (0.25 gram) daily was adequate to cure well-advanced cases of xerophthalmia, the eye disease which develops when vitamin A is absent from the diet. J. S. H.

NOTES FROM THE U. S. BUREAU OF MINES.*

TEMPERATURE CONTROL SYSTEM FOR DRESSING AND TEMPERING FISHTAIL BITS.

By Charles H. Shapiro.

THE Bureau of Mines, through its Petroleum Experiment Station, has conducted a field study of methods used in oil-field shops for dressing and tempering fishtail bits, in order to develop practical methods of accurate temperature control, thus doing away with the source of most of the trouble experienced in the present method of gaging temperatures by eye.

Experiments were conducted on a new method devised by a shop in the Stroud, Okla., oil field. The method accurately regulates, by the use of a bath of molten salt, the temperature to which a bit is heated before being quenched to produce hardness.

The system is made up of a suitable salt mixture contained in a large rectangular cast-steel pot, together with a pyrometer and thermocouple for measuring the exact temperatures to which the salt is heated. The steel pot is ¾ inch thick, 36 inches long, 6½ inches wide, and 8 inches deep, having a flange 3½ inches wide around the top. The steel pot is set into a furnace suitably built of fire brick and so arranged that it is heated from underneath. A layer of compounded salt three inches thick is placed into the pot, and heated to the temperature corresponding to the critical temperature of the steel being hardened, as indicated upon the pyrometer. Bits to be tempered are then placed in the molten bath, and quenched. The use of molten salt serves no purposes other than to act as a medium for uniform heating, and as a means whereby a pyrometer can be used to measure exact temperatures. There is no reaction between the salt and the bit.

A salt bath system similar to the one at Stroud was later set up in the Tonkawa field. Exceptionally good results were obtained with bits treated in the new way, but considerable trouble was experienced with the performance of the system itself. The long and continued heating of the salt caused the formation of a heavy sludge which would drop to the bottom of the pot. This sludge

* Communicated by the Director.

tended to retard the passage of heat, concentrating it upon the bottom of the pot. In order to keep the salt at the required temperature, a hotter flame was found necessary. The concentration of heat upon the bottom of the pot would slowly burn it, thereby causing a leak.

After numerous experiments with various salt mixtures, it was decided to substitute lead in place of salt as the heating medium. Lead has a lower melting-point, a much higher heat conductivity, and because of its high specific gravity (weight compared with an equal volume of water) impurities will not settle to the bottom of the pot and retard the heat, but instead will float upon the surface, where no harm can be done. Unless the lead is impure or oxidized, it will not adhere to the bit nor will it have any chemical effect on the bits or upon the steel pot in which it is heated. Because of its higher heat conductivity, bits will heat faster. It has also the property of reaching the necessary temperature faster and of holding that temperature with a much smaller flame underneath the pot than is required with the salt mixture. Molten lead has a tendency to oxidize when in contact with air, forming lead dross. This can be eliminated by putting a small amount of compounded salt into the pot with the lead. When heated, the salt will melt and float upon the surface of the lead, excluding the air. The layer of salt being somewhat colder than the molten lead, any sharp line between the heated and unheated portion of the bit may be eliminated by varying the thickness of the salt layer. The sudden shock due to immersing a cold bit into the hot lead is very greatly decreased because of the film of salt that will coat the bit when first set in. A film of salt will also adhere to the bit upon being removed, thus preventing oxidation.

It has been found that the time necessary to temper a bit by the new system is approximately the same as the time required by the old. The labor expended in tempering is considerably less than in the old system, hoisting and moving about being all done by mechanical means (crane and hoist). Because of the difference in construction of the furnaces, it is found that the men do not have to work in such terrifically hot atmosphere as is the case when tempering by the old system.

The figures obtained in this investigation show that a proper method of heating and quenching fishtail bits will increase the

footage obtained per bit by more than 50 per cent. and reduce the drilling time per foot by a third of that obtained when bits are treated in an open fire without temperature control and quenched under conditions that vary from time to time. Further details are presented in Serial 2712.

GASES IN STREET MANHOLES.

By S. H. Katz, E. G. Meiter, and J. J. Bloomfield.

Street manholes for underground power and signal circuits, and for sewers vary in size from small openings that a workman can enter, to large central vaults.

The writers tested the composition of atmospheres in some telephone-circuit manholes at Pittsburgh, Penna., and Philadelphia, Penna. They are indebted for arrangements permitting the tests in manholes at Pittsburgh to F. K. Singer, division superintendent of methods, and W. M. Dickson, division supervisor of safety, of the Pittsburgh Division of the Bell Telephone Company of Pennsylvania; for assistance at the manholes to S. J. Rowe; for arranging the tests at Philadelphia to A. B. Detwiler, division results supervisor, and W. M. Day, safety engineer of the Philadelphia Division of the Bell Telephone Company of Pennsylvania.

The atmospheres in fifteen manholes at Pittsburgh, Penna., and twelve at Philadelphia, Penna., suspected of being gaseous, were tested. Some of the holes contained city gas which had leaked into the holes from distributing mains. The natural gas distributed at Pittsburgh is non-poisonous, provided no artificial gas is added, but large volumes may suffocate workers in unventilated manholes by displacing the air. Manufactured fuel gas distributed at Philadelphia is high in carbon monoxide, which is poisonous; small amounts render the air in manholes unfit to breathe.

Gassy manholes require thorough and continuous ventilation to insure safety to workers. A hand-operated rotary blower, similar to a forge blower, with canvas pipe leading to the bottom of the hole is serviceable.

Tests were made for CO with palladium chloride test papers, which change from yellow to gray or black when moistened and exposed to 0.25 per cent. or more of CO for five minutes; this

simple test shows presence of very dangerous amounts of CO such as will quickly overcome a man, but it is not sensitive enough to indicate lesser amounts which will affect a man more slowly. The iodine pentoxide or hoolamite indicator for CO gave color changes from light gray to bluish-gray when 0.07 per cent. or more CO was present, and is a better indicator than the test papers, for CO in concentrations which may affect men.

Tests for explosive gas mixtures were made at the holes with a gas bomb constructed of a 2 by 2 by ¾-inch pipe tee, reducers, stop-cocks, and an automobile spark-plug in the ¾-inch opening. Gases from the hole were transferred to the bomb with a rubber hand bulb and tubing, and sparked by means of two dry cells and an automobile spark coil. Explosions were indicated by a kick on a pressure gage on the bomb.

Explosive gas mixtures were found in one hole at Pittsburgh and two at Philadelphia.

The hose mask is recommended for use in manholes when respiratory protection is necessary. Additional information on the methods of testing and on the results of the tests is presented in Serial 2710, recently issued by the Bureau of Mines.

Test for Diethyl Phthalate.—SAMUEL LEVINSON (*Ind. and Eng. Chem.*, 1925, **17**, 929) recommends the following procedure for the detection of the presence of diethyl phthalate in alcoholic liquids. A 10-c.c. sample of the liquid is concentrated to a small volume by heating over a free flame; 1 gram of dry boric acid is added; and the liquid is evaporated to dryness. Then 1 gram of resorcinol is added; and the mass is heated until it becomes sticky and almost dry. After cooling, the mass is dissolved in 30 c.c. of water with the aid of heat. The resulting solution is transferred to a tall glass cylinder, and is made alkaline by addition of sodium hydroxide. A fluorescence indicates the presence of diethyl phthalate. J. S. H.

Effect of Long-continued Cold-storage on the Vitamin A Content of Frozen Eggs.—D. BREESE JONES, JOSEPH C. MURPHY and OTTO MOELLER (*Am. Jour. Physiol.*, 1925, **71**, 265–273) have determined the vitamin A content of hen's eggs which had been removed from the shell, thoroughly mixed, and preserved hard frozen in cold-storage for a period of almost nine years. In daily doses of 0.25 gram, the storage eggs were fully as effective as fresh eggs in curing xerophthalmia in albino rats. It is possible that a slight, but practically insignificant, deterioration in vitamin A content occurred during the prolonged cold-storage. J. S. H.

THE FRANKLIN INSTITUTE.

THE stated meeting of the Institute was held at eight-fifteen o'clock on the evening of Wednesday, October twenty-first, 1925, in the Hall of the Institute. President W. C. L. Eglin presided.

The President announced that the minutes of the last stated meeting had been printed in the JOURNAL OF THE INSTITUTE for June, and that unless there were objections, they would be approved as printed. No objection was offered and the minutes were therefore declared to be approved.

The Secretary read the following statement on behalf of the Board of Managers:

> The Board of Managers recommends the establishment of a class of members to be known as Student Members. In order to bring about this result, the Board gives notice at this meeting that at the stated meeting of the Institute to be held two months hence, it will move the adoption of the following article as a substitute for Article II—Members, of the present By-Laws of the Institute.

Section 1. Members of the Institute shall consist of those engaged or interested in scientific pursuits or in the application of science in the mechanic and industrial arts. All persons interested in the purposes and activities of the Institute, and who are willing to further them, may become members when proposed by members in good standing and elected by the Board of Managers. The membership of the Institute shall consist of the following classes of members, *viz.*, Honorary and Corresponding, Endowment, Life, Contributing, Resident, Non-resident, Student, and holders of Second-class Stock.

Section 2. Honorary and Corresponding members shall be nominated by the Board of Managers and shall require for their election four-fifths of the votes of the members present at any stated meeting of the Institute at which their nomination may be acted upon. They shall not be entitled to vote nor to hold office. All other members shall be elected by the Board of Managers.

Section 3. Endowment members shall consist of persons, firms, corporations or associations who shall make an endowment payment of five thousand dollars ($5000) to the Institute, and who, upon acceptance thereof by the Board of Managers, shall thereafter have the privilege of nominating annually to the Board of Managers for election (subject to its discretion as to any particular nominee) as Resident members of the Institute for its then current year, without payment of dues, that number of persons, to be determined from time to time by the Board of Managers, whose annual dues, if they were paying annual Resident Membership dues, would approximately equal but not exceed the then current income from such endowment payments. Such Endowment Memberships shall be perpetual, and shall be transferable by the holders thereof by will or otherwise; provided, however, that the Board of Managers at any time may refund five thousand dollars ($5000) to the then holders of any such membership, and annul and terminate that membership.

Section 4. Contributing members shall consist of firms, corporations, associations or individuals who shall pay annually the sum of three hundred dollars ($300). A contributing member shall have the right to nominate nineteen persons to the Board of Managers for election as Resident members for the year then current, subject to the discretion of the Board as to any particular nominee, and members thus elected shall pay no dues.

Section 5. Resident Life members, whose membership shall not be transferable, are those members who shall pay the sum of three hundred dollars in any one year.

Non-resident Life members shall be those members who reside permanently at a distance of not less than twenty-five miles from Philadelphia and who shall pay the sum of one hundred dollars in any one year.

Section 6. Resident members shall be residents of the City of Philadelphia who have been elected to membership in the Institute and who shall pay annual dues of fifteen dollars.

Section 7. Non-resident members shall be those members who reside permanently at a distance of not less than twenty-five miles from Philadelphia and who shall pay annual dues of five dollars. They shall also pay an initiation fee of five dollars.

Section 8. Student members shall be over sixteen and under twenty-two years of age. They shall pay annual dues of three dollars if they do not receive the JOURNAL OF THE INSTITUTE, and six dollars if they do receive the JOURNAL. The term of student members shall be limited by the age of twenty-two years, after which time they shall be eligible for transfer to the other types of membership upon the payment of the appropriate fee or dues.

Section 9. Privileges: All members, of whatever class, are entitled to participate in the meetings of the Institute, in the use of the Library, to receive tickets for the lectures, to attend section meetings, and to receive monthly one copy of the JOURNAL OF THE INSTITUTE, except that student members, and honorary and corresponding members shall not have the right to vote, nor to hold office, and student members paying the three-dollar annual dues shall not receive the JOURNAL OF THE INSTITUTE.

The Secretary reported for the information of the Institute that since the last stated meeting there had been a gain in membership of seventy-six, with a loss from various causes of thirty-six, leaving a net gain of forty members for the period mentioned.

The lecture of the evening was delivered by Prof. Edwin G. Conklin, Professor of Biology in Princeton University, on the subject, "Why Teach Evolution?" Professor Conklin spoke very entertainingly and brilliantly. His lecture was followed by a brisk discussion, in which five members of the audience participated, together with Professor Conklin.

A vote of thanks to Professor Conklin for his stimulating and enlightening lecture, offered by Mr. Penrose, was unanimously adopted by a rising vote.

The meeting then adjourned, at nine-thirty-one P.M.

HOWARD MCCLENAHAN,
Secretary.

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, October 7, 1925.)

HALL OF THE INSTITUTE,
PHILADELPHIA, October 7, 1925.

MR. HAROLD CALVERT *in the Chair.*

The following report was presented for final action:

No. 2810: Van Sicklen Chronometric Tachometer.

This instrument was devised for the purpose of determining accurately the speed of a rotating shaft or the surface speed of a rotating wheel. The speed of a shaft can be determined by the combined use of a cyclometer and stop-watch, but the accuracy of the determination depends upon the starting and stopping of the two instruments at exactly the same time. The chronometric tachometer, as the name indicates, is a device that includes in a single instrument a measurer of both the time unit and the rotations for that time.

Placing the driving point in contact with the centre of the shaft starts both the watch and counting mechanisms, the latter controlling the position of a pointer. At the expiration of the time unit both movements automatically stop, the pointer being locked in position and the driving mechanism returning to its initial position.

The original form of this device was developed during the time of the World War, while the present form is an improvement that includes a simplification of construction that adapts it to modern methods of quantity production.

United States Patent No. 1,447,426 was granted for this device on March 6, 1923, and on the recommendation of the Committee the Certificate of Merit was awarded to the inventors, J. C. and W. A. Prouty, of California, for designing a high grade of Chronometric Tachometer that is adaptable to modern methods of quantity production.

The following report was presented for first reading:

No. 2821: Gray Sheet Metal Cutter.

GEO. A. HOADLEY,
Secretary to the Committee.

MEMBERSHIP NOTES.

ELECTIONS TO MEMBERSHIP.

(Stated Meeting, Board of Managers, October 14, 1925.)

RESIDENT MEMBERS.

MR. WILLIAM LIKENS BROWN, District Manager, Industrial Works of Bay City, Michigan, 1505 Packard Building, Philadelphia, Penna.

MR. ALFRED PERKINS LOHMANN, Engineer, 15 Railroad Avenue, Haverford, Penna.

MR. HERBERT L. TOWLE, Technical Advertising, 4553 Adams Avenue, Frankford, Philadelphia, Penna.

MR. CHARLES E. TULL, Electrical Contractor, 1832 Ludlow Street, Philadelphia, Penna.

NON-RESIDENT MEMBERS.

MR. BENJAMIN FRANKLIN ADAMS, Vice-president, Charles F. Adams Company, 632 West Chestnut Street, Lancaster, Penna.

MR. VICTOR G. BLOEDE, Manufacturing Chemist, Carroll Post Office, Baltimore, Md.

MR. J. E. EGLESON, Chemical Engineer, General Chemical Company, 40 Rector Street, New York City, N. Y.

MR. JOHN J. HARMAN, Mechanical Engineer, in care of Walworth Manufacturing Company, Lock Box 26, Boston, Mass.

MR. Y. MATUNAGA, Physicist, The Research Laboratory, Shibaura Engineering Works, Kanasugi, Shiba, Tokyo, Japan.

MR. CHARLES H. NORDHAUS, Radio Experimenter, 4313 North Springfield Avenue, Chicago, Ill.

DR. PAUL E. SABINE, Physicist, Riverbank Laboratories, Geneva, Ill.

MR. WILBUR ANDERSON SMITH, Student, 808 Station Avenue, Haddon Heights, N. J.

CHANGES OF ADDRESS.

MR. MERRITT H. ADAMS, Rainer Trucks, Inc., Fifth and Webster Avenues, Long Island City, N. Y.

DR. ROBERT J. ANDERSON, 1658 Woodward Avenue, Lakewood, Cleveland, Ohio.

MR. CHARLES A. D. BURK, 21 South Oakland Avenue, Ventnor, N. J.

LT.-COL. EDWARD B. CLARK, Potomac Hotel, Washington, D. C.

MR. J. C. DACOSTA, 3D, Indemnity Insurance Company of America, 212 South Third Street, Philadelphia, Penna.

PROF. WILFRED T. DAWSON, Medical Department, University of Texas, Galveston, Texas.

MR. PAUL DIDIER, Maloney Building, 339 Second Street, Pittsburgh, Penna.

MR. CLARK DILLENBECK, 209 Pelham Road, Mt. Airy, Philadelphia, Penna.

MR. JOHN V. N. DORR, 247 Park Avenue, New York City, N. Y.

MR. ARTHUR FALKENAU, 4 East Eighty-eighth Street, New York City, N. Y.

MR. E. W. FINKBINER, 6007 Oxford Street, Philadelphia, Penna.

MR. W. F. GRAHAM, Chief Metallurgist, American Nickel Corporation, Clearfield, Penna.

MR. RICHARD GREENWOOD, 4520 Worth Street, Frankford, Philadelphia, Penna.

MR. ANTHONY DEH. HOADLEY, Union College, Schenectady, N. Y.

MR. S. C. HOLLISTER, 311 Elm Avenue, Swarthmore, Penna.

MR. ERNEST L. HUFF, 517 Coleman Avenue, Fairmont, W. Va.

MR. W. N. JENNINGS, Smith Building, Eighteenth and Market Streets, Philadelphia, Penna.

MR. CHARLES F. KELLERS, 117 North Terrace, Schenectady, N. Y.

MR. J. C. LYNCH, Assistant Vice-president, American Telephone and Telegraph Company, 195 Broadway, New York City, N. Y.

MR. THADDEUS R. PARRISH, 236 South Oak Park Avenue, Oak Park, Ill.

MR. ROWLAND S. POTTER, 25 Trinity Place, Montclair, N. J.

MR. C. H. QUINN, Room 420, Pacific Mutual Building, 660 Market Street, San Francisco, Calif.

MR. RICHARD G. SAGEBEER, 9 Dunstable Road, Cambridge, Mass.

MR. A. C. SHAND, Merion, Penna.

MR. H. B. SPENCER, 2012 Massachusetts Avenue, Washington, N. W., D. C.
PROF. V. A. SUYDAM, 1224 Porter Avenue, Beloit, Wis.
MR. H. W. WELLES, 224 West Fisher's Avenue, Olney, Philadelphia, Penna.

NECROLOGY.

Mr. Wilson Catherwood, 1708 Walnut Street, Philadelphia, Penna.
Dr. Charles Frederick Chandler, East Hartford, Conn.
Mr. John M. Diven, Superintendent, The Water Works, Troy, N. Y.
Mr. E. C. Fisher, 308 North Jefferson Avenue, Saginaw, Mich.
Dr. S. Peacock, Wheeling Steel Building, Wheeling, W. Va.
Mr. Carl P. Schlicke, 191 Berkeley Place, Brooklyn, N. Y.

LIBRARY NOTES.

RECENT ADDITIONS.

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ANDERSON, ROBERT J.—Metallurgy of Aluminium and Aluminium Alloys. 1925.
ARRHENIUS, S. A.—Chemistry in Modern Life. 1925.
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BRAGG, SIR WILLIAM.—Concerning the Nature of Things. 1925.
CADY, F. E., and DATES, H. B.—Illuminating Engineering. 1925.
Chemical Engineering Catalog. Tenth annual edition. 1925.
COHEN, J. B., and RUSTON, A. G.—Smoke: A Study of Town Air. New and enlarged edition. 1925.
FISCHER, FRANZ.—The Conversion of Coal into Oils. 1925.
FOSTER, WILLIAM.—The Elements of Chemistry. 1925.
Gesammelte Abhandlungen zur Kenntnis der Kohle. Vol. 7. 1925.
HAMLIN, M. L., and TURNER, F. M.—The Chemical Resistance of Engineering Materials. 1923.
HODKIN, F. W., and COUSEN, A.—A Textbook of Glass Technology. 1925.
HUYGENS, CHRISTIAAN.—Oeuvres Complètes. Vol. 15. 1925.
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International Congress for Applied Mechanics, Delft, 1924. Proceedings. 1925.
PEARSON, WILLIAM A., and HEPBURN, JOSEPH S.—Physiological and Clinical Chemistry. 1925.
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Poor's Industrial Section, 1925. Vol. I. 1925.
Poor's Public Utility Section, 1925. 1925.
SENNETT, RICHARD, and ORAM, SIR HENRY J.—The Marine Steam Engine. Fourteenth edition. 1924.
TRINKS, W.—Industrial Furnaces. Vol. I. 1923.

BOOK REVIEWS.

THE ELEMENTS OF CHEMISTRY. By William Foster, A.M., Ph.D., Princeton University. xviii–576 pages, over 200 illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $2.

Textbooks of elementary chemistry are quite numerous. Each teacher seems to be dissatisfied with the works of others. This is not unnatural, for there is probably no science in which there is such great difficulty in starting and continuing instruction as chemistry. The science has two phases: Theory and experiment. If we contrast the task of the professor of elementary chemistry with that of the teacher of biology or its specialties, zoölogy or botany, we may see the comparative easiness of the natural history studies. The teacher of botany has no difficult theory to explain. Even the theory of evolution, now so much discussed, may be wholly left out. By proper study and investigation any intelligent person may become familiar with the *Orobanchaceæ* without giving a thought to possible descent from the *Scrophulariaceæ*. In chemistry, however, complex theories involving now, in some phases, higher mathematics, are taught and then students must be guided through a lot of experiments, mostly requiring fair manual skill and not infrequently dangerous unless strictly carried out. Writers of textbooks have differed very much in their method of approach to the general principles and elementary facts of the science. Some prefer to lay down the theoretical data at once, following immediately with principles of notation and nomenclature, data in regard to salt formation, electrolysis and allied general phenomena, and then take up each element in some connected order. The usual method in the United States seems to be to describe the properties and principal compounds of the so-called "non-metallic" elements, then the metals proper, after which the organic chemistry is taken up. Some teachers, however, distribute the theoretical teaching in admixture with the descriptive portion. The old "Fownes," which was a very popular textbook sixty years ago, treated the non-metals with almost no reference to formulas or theories, and then devoted much space to these latter subjects before proceeding to the description of the metallic elements and their compounds.

The work in hand distributes the theoretical dosage through the descriptive text. It contains a large amount of information carefully and accurately expressed and takes rank with the standard elementary manuals. Evidently great care has been taken in the selection and arrangement of the data, and the author's experience has been comprehensive and extended. The reviewer differs with the author as to the advisability of certain insertions. A considerable number of illustrations are devoted to representation of minerals and masses of metals. Most of these are mere smudges and can certainly give the student no proper idea of the substance. There is but one way to teach the appearance and general properties of any substance, that is by inspection. Pictures of a silver tree and of a mass of native copper are not worth the cost of the photograph and the reproduction. In the table of elements with symbols and atomic weights, the atomic numbers are also given, but as the elements are arranged (properly) alphabetically, the atomic numbers are jumbled up so that the impressiveness of their sequence and the fact that

certain gaps occur are lost. This sequence arrangement is all the more important in view of the probable discovery of three elements which fill the gaps. Hafnium, rhenium, and masurium have come too late for incorporation into the book, but any teacher using it would be aided if he could call attention to the probable relations of these to Moseley's suggestions. In noting the discovery of helium, reference is made to Jansen's observations in 1868, but the usual bibliographic note of this is an error, as Jansen does not mention the observation of the special line. By a slip the subsequent allusion to helium is credited to Lockyer and Franklin, Frankland being, of course, meant. Errors such as these will creep into any manual. They are merely the inevitable manifestations of human fallibility. Undoubtedly, students using this book will get a comprehensive and excellent knowledge of chemistry, but it is the reviewer's experience, after many years of teaching, that the lecture system is the best for instruction in chemistry. The low price of the book is commendable.

HENRY LEFFMANN.

THÉORIE NOUVELLE DE LA PROBABILITÉ DES CAUSES. Par Stanislas Millot. vi–36 pages, 6½″ × 10″, paper. Paris, Gauthier-Villars et Cie., 1925. Price, 5 francs.

Since the day of Fermat and Pascal, when the active investigation of the theory of probability may be said to have had its beginning, there are few mathematicians of renown whose names are not associated with its development. Yet the theorem of that illustrious early investigator, James Bernoulli, survives the test of time and is still, with its acknowledged limitations, in current use.

The present essay deals with the inversion of the Bernoulli theorem which is, that in a large number of trials m in each of which there is a probability p, that an event will occur, there is nearly enough a probability

$$P = \Theta(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx \qquad (1)$$

that the deviation from p shall not exceed in absolute value

$$l = x \sqrt{\frac{2p(1-p)}{m}}. \qquad (2)$$

As the author explains, p may be the unknown, and if a is the number of times the event occurs in the m trials, it is usual to take $\frac{a}{m}$ as an approximate value of p and then proceed tentatively by the *a priori* method till consistent results are obtained. To avoid the uncertainty of this method, he substitutes in (2) the variable y in the place of p, of which it is a particular value. Then the deviation l may be written

$$\frac{a}{m} - y \leqq x \sqrt{\frac{2y(1-y)}{m}}. \qquad (3)$$

By reference to the graph of this equation between x and y he derives formulas for direct *a posteriori* computation of statistical inference, where the probability p is the quantity to be determined. A number of examples illustrative of the

simplicity of the application is discussed. Among these are some taken from the classic treatise of Laplace with which they are in substantial agreement. The title, which is a bit misleading, is doubtless adapted from that of a chapter in Laplace, "On the Probability of Causes and Future Events Inferred from Observed Events." The work is a revised reprint of a report in *Comptes Rendus* sponsored by the well-known author of the now widely used Monograms or "Parallel Coördinates," M. Maurice d'Ocagne.

The method which the author has employed is ingenious, direct, and appealing to the practical user, and is suggestive of further application in clarifying other difficulties in the field of probability. LUCIEN E. PICOLET.

PHYSIOLOGICAL AND CLINICAL CHEMISTRY. By William A. Pearson, M.D., Ph.D., Department of Chemistry, Hahnemann Medical College of Philadelphia, and Joseph S. Hepburn, A.M., Ph.D., Associate Professor of Chemistry in Hahnemann Medical College of Philadelphia. 306 pages, illustrated, interleaved, small 8vo. Lea and Febiger, Philadelphia and New York, 1925. Price, $4.

This book, like others that cover the same field, makes a striking contrast to the chemistries offered to medical students in the middle of the last century. American medical colleges then exacted no preliminary educational requirements. The professor of chemistry was obliged to start with the simplest data and build up slowly until he could reach in the closing days of the course a few demonstrations on clinical subjects, almost entirely limited to the tests for urine. It is ungracious to sneer at the sins of commission and omission of our forebears, for they built as wisely as they knew and labored with far more limited means than are now available. Yet one cannot but feel the force of Wurtz's criticism of the old method of teaching organic chemistry, when, as he says, the course began with the sugars and starches, substances about which the structural formulas were unknown, proceeded to the study of a group called the vegetable acids, because they have the common property of turning blue litmus red, and later the group called alkaloids because they have the property of turning the red litmus back to blue.

The book in hand is a comprehensive and clear summary of the more important tests and procedures as required at the present time in the application of chemistry to clinical problems. The authors have had large experience both in carrying out such work and in teaching the methods to students, and they set forth the results of such work in compact text. The preface informs us that the book covers a course, to which students are admitted after considerable preliminary instruction in chemistry, which is the standard now of all reputable medical schools in this country. In consequence of this preliminary training, chapters on qualitative and quantitative analysis are merely brief summaries of the topics. The interleaving allows of note-taking by the student and also of any corrections or of changes in detail of methods as are constantly appearing in periodical literature.

The reviewer is compelled to express disapproval of the long list of titles attached to each author's name. Both the authors are quite well known as excellent chemists, their degrees and the fact that they are occupying responsible positions in a first-class medical school will be sufficient recommendation

if any is needed. Membership, for instance, in the American Chemical Society is proper, but as a matter of fact it is no professional recommendation for the door to it is wide open.

The student in physiologic and clinical chemistry will find in this book all the important information on the subject. On one point the reviewer questions the appropriateness of the test chosen, namely, in the detection of methanol. The old copper spiral method is given. The test by oxidation with permanganate, with the modification suggested by LaWall, now official in U.S.P.X, should have been described. HENRY LEFFMANN.

A TEXTBOOK OF GLASS TECHNOLOGY. By E. W. Hodkin, B.S., A.I.C., and A. Cousen, M.S., A.R.C.S., A.I.C. With a foreword by Professor Turner, of the Department of Glass Technology, University of Sheffield. xxiii–551 pages, 251 illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $12 net.

It cannot be said of this book that the industry to which it is devoted is seen "through a glass darkly," for it is a most explicit and comprehensive presentation of the whole field of glass manufacture, theory as well as practice being discussed. One gets from it an impressive idea of the importance of glass in human affairs. In the foreword by Professor Turner, it is stated that works in English on the subject are few. Books on stained glass and manuals for collectors are not lacking. Such books, indeed, find large sale; publishers are not afraid to make great outlay in the printing of them, for those who buy are accustomed to paying high prices for what they wish, but a work on the technology of glass appeals to an entirely different public. Much credit is due the publisher for the excellent manner in which the book has been got up, many of the photogravures being most commendable in clearness and brilliancy. Especially commendable is the plate facing page 5, showing several fine pieces of clear glassware made in England in the first half of the eighteenth century.

The authors give as an introduction a highly interesting account of the development of glass-making from the earliest known state to the present. The story told in classical history of the accidental discovery of the method of making glass is not now looked upon with favor. The earliest definite manufacture is assigned to Egypt. According to Petrie, this manufacture in Egypt, as distinct from the glazing of pottery, dates about 3500 years ago. The glass factories of Alexandria became famous. Phenician traders probably carried the products to many parts of the Mediterranean borderlands. Rome received much glass from Alexandria, but about the beginning of the present era, Egyptian glass-workers found their way to the Eternal City and in the first quarter of the third century of the era, Alexander Severus found the industry so well developed that he levied a special tax on it. The separation which then existed between the scientists and philosophers, on the one hand, and artisans and tradesmen on the other, prevented the coöperation of theory and practice that is so characteristic of the present age.

American readers and students will be interested in the note of the development of the glass industry in the Western Hemisphere. This, it appears, was a factory for the manufacture of window glass, established in 1790, by Robert Hewes in New Hampshire. It was not successful and the industry languished

for some years. About the beginning of the nineteenth century the industry took a new life and succeeded. All types of glass manufacture and glass-working were, of course, originally carried out by manual skill with only limited machinery available. Of recent years, machines have been extensively applied, especially in bottle-making. Here American ingenuity leads. Semi-automatic machines were first devised, but now wholly automatic apparatus is available, and the bottle-making machines employed by some of the large bottling companies are entirely satisfactory. Judging by what one sees in the stores and scattered about the open lots and streets, bottle-making in the United States must be one of its most extensive industries. The book naturally devotes the principal space to the more extended phases of the industry, but the numerous special applications of glass, such as ornaments, mirrors, and lenses are given careful consideration.

Though intended as a textbook it is not an elementary treatise, for the technical language of the industry is constantly in evidence. To the chemist who is only superficially acquainted with the manufacture of glass, the array of unfamiliar terms is somewhat disconcerting. Such terms as "lehr," "marver," and "parison" indicate the wide range of languages from which the phraseology of the industry has been drawn. All phases of the glass manufacture as commonly understood are presented, including pyrometry. The testing of the products for resistance to ordinary atmospheric influences and to common chemical agents is set forth. In the latter connection, mention is made of the now very popular "pyrex," which has replaced largely the German glass in chemical laboratories. This, as is the case with the older resistance glasses, is a borosilicate.

In looking through this volume, admirable both as to form and contents, there is suggested a thought as to how much of the glass industry may before long give way to the manufacture of quartz glass. This manufacture is, it is true, restricted by cost and the limited applications of the product, but the progress of invention may eliminate the former and with cheapening, the applications may be greatly extended. Chemists will certainly find much use for quartz glass if it can be obtained at moderate price, and the peculiar properties of it in relation to ultra-violet radiation bids fair to extend its use greatly in optical instruments. The high cost of platinum has also given an impulse to its use. It seems that quartz glass for skylights has a specific value in the treatment of some diseases.

No doubt a very careful scrutiny of the text would show errors, typographical or in form, but the text is in very satisfactory condition. On page 210, third line, "silver-constantin" seems to be incorrectly given as containing copper.

HENRY LEFFMANN.

THE METALLURGY OF ALUMINIUM AND ALUMINIUM ALLOYS. By Robert J. Anderson, B.Sc. xxxi–930 pages, numerous illustrations, bibliography, 8vo. New York, Henry Carey Baird Company, 1925. Price, $10.

This extensive and well-printed volume contains an immense amount of information about a metal that has comparatively recently taken a position in industrial life. The reviewer remembers an ingot weighing about an ounce that was a curiosity in the chemical collection of the Central High School of Phila-

delphia, about 1860, having been brought from France at a cost of about five dollars. For many years this was all the part that the metal played in chemistry or industry. Mr. Anderson traces the development of its metallurgy from the days of H. St. Claire-Deville to the present time when the production has become an enormous industry and the metal has found application in many ways. His large experience in all phases of this industry and in all the ramifications of the pure metal and its alloys insures the production of a valuable treatise. There was need for such a book, for Richards' work has long been out of print, and the metallurgy of aluminum has grown far beyond the limits of that excellent and classical treatise. It will be noted that Mr. Anderson uses the spelling that has unfortunately been adopted by the American Chemical Society, instead of the form "aluminum" which not only saves a letter, but makes the term for the oxide, "alumina," more congruous. However, the efforts to simplify English spelling have practically gone "glimmering." Ephraim is joined to his idols and must be let go until the spirit of reform again arises. The book is dedicated to the memory of Joseph W. Richards, and is a worthy tribute to him.

Aluminum is one of the most abundant metals, but the difficulties of extraction kept it a mere curiosity until the introduction of improved processes, especially electrolytic methods, brought its cost to such a point that it could compete with the common metals. A table shows the fall in price. In 1852 the price was considerably over $500 per pound. By 1891 it had fallen below a dollar and has been below a half-dollar since 1895. In 1911 and 1922 it reached the lowest price noted in the table, twenty cents per pound. It has naturally found extensive application in the automotive industries, on account of its combination of lightness with tensile strength. Its alloy with several metals, of which copper is the dominating component, known as duralumin, has found a wide application. It has the property of hardening on quenching followed by ageing or maturing. Magnalium, an alloy with magnesium, is very light and strong and is used for beams in analytical balances. It is a curious fact that this alloy has a specially high reflective power on ultra-violet rays, being better than silver. In the early days of reduction of price of aluminum, an alloy with copper having a brass color and much lustre was employed in the manufacture of cheap jewelry under the name "Milton gold," but Anderson states that these bronzes have but little metallurgic importance. In connection with the earliest winning of aluminum, the question of Oersted's claim is discussed briefly, Wöhler being given the credit of having first prepared the metal. Some recent publications seem to place Oersted's claim upon a more secure basis than has heretofore been supposed.

It is impossible in the space available for a review to give even a superficial summary of the contents of this admirable volume. Here will be found the essence of all that is known in this great industry, an industry that is now comparable in its relation to modern life with that of iron. Aluminum has entered into so many phases of our daily life that one feels puzzled as to how the world got along without it. Not the least of the applications is that for the manufacture of kitchen utensils. Aluminum ware is a feature of the retail stores. One of the most striking applications is in the manufacture of thermit, the mixture which has proved so useful in repairing large castings, and for

other welding processes. The finely powdered metal figures in paints for iron work, though it has been shown by the Bureau of Standards that painting the ordinary house radiator with such paint is not economical, as it interferes with radiation according to a well-known law. Aluminum dust is highly explosive when mixed with air. A violent explosion attended with loss of life and property occurred some years ago in a factory in which aluminum kitchen-ware was being polished. The fine powder has been used in photographic flash powders.

The book contains nearly three hundred excellent illustrations, among which are many photomicrographs of the metal and its alloys, to show the intimate structure as is now so frequently done in metallurgy. We have in short in this book a complete story of one of the most interesting and important contributions to the metal resources of the world by chemical research.

HENRY LEFFMANN.

CHEMISTRY IN MODERN LIFE. By Dr. Svante August Arrhenius. Translated from the Swedish by Dr. Charles Shattuck Leonard. xvi–286 pages, small 8vo, illustrated with plates and figures in the text. New York, D. Van Nostrand Company, 1925. Price, $3.

This work is part of a series of the "Library of Modern Sciences" issued under the editorship of Messrs. Slosson, Luckiesh and Howe. Works intended to popularize chemistry have been in vogue for many years. The present reviewer had his attention drawn to the science in the early sixties of the last century by articles in a journal for working-men, for which one of the older members of the family was a subscriber. Several comprehensive and excellent books have appeared in the last few years. Earlier efforts were devoted merely to bringing to general notice the interesting features of the science and explaining to some extent its more difficult data, but of late a missionary element has developed, in the endeavor to impress upon the English-speaking nations the importance of research as a means of protection against a militant spirit. The lessons of the late war have not been thoroughly learned by either British or American peoples. The coöperation of theory and practice that has characterized German science does not find an entirely hearty echo in America, although considerable improvement has taken place of late years.

The translator's preface is a discussion of this subject: "Here in America we sit back and beam about us on a land of plenty and resource. Of course, we are bothered by a few petty things—next winter's coal, the cost of electric power for that new factory project, or the rising cost of construction steel—but we lay these evils upon the miner, or the service company or the steel trust and continue our serene way." The author also stresses the increasing pressure of our industrial civilization. He says that physics and chemistry have revolutionized our daily life. Revolutions are by no means always changes for the better, and it may be a question whether the modern conditions with their smoke, noise, congestion of population and of street transit, and the demand for labor bringing about intermingling of races substantially incongruous, are improvements upon the simpler life of fifty years ago. Will chemistry come to the rescue? Probably it will give us new poisonous gases, higher explosives,

brighter dyes, and new synthetic drugs to run their brief course of enrichment of German manufacturers.

Unlike most of the works intended to popularize chemistry, Doctor Arrhenius' book begins with a short history of the early development of science, or at least the ancient ideas of the constitution of matter. The views of some oriental cults concerning the nature of creation are set forth, but the "conflict" that is now raging in the United States is dextrously avoided. Chemists, indeed, do not need to enter into this controversy. The science is purely materialistic. So far as the general nature of the book is concerned, it is to be said that it contains a large amount of information expressed in clear and concise form. The translator has done his work well, the text showing no strange or uncouth phrases, as may be the case with renditions from a foreign language in carelessness or ignorance. The translator makes an urgent appeal for conservation. It is to be feared that his voice and that of many others raised in the same cause will be without avail. Ominous utterances are coming out of the western part of the United States, from prominent politicians, indicating an organized and determined raid upon the conservation program of the general government concerning forests, water-power, oil lands and other important reserves.

Doctor Arrhenius is one of the most prominent chemists of the day and deservedly so. His work has been epoch-making, "bahn-brechend" as the Germans would say. In this book he has set forth many facts of interest and value to the general reader, and it is to be hoped that it will find a large recognition. Some of the illustrations are rather sketchy, but there are several excellent ones and also plates. It is doubtful if in a work of this kind any use should be made of formulas. On page 47, the formula for phosphine is stated to be PH_3 and to mean thirty-one parts of phosphorus and three of hydrogen. This is true, but might not such a statement lead one ignorant of the true meaning of formulas to assume that PCl_3 means thirty-one parts of phosphorus and three of chlorine? It is gratifying to note that Robert Boyle is given credit for making one of the most important advances towards the establishment of scientific chemistry. Doctor Arrhenius seems not to be under the spell of the Lavoisier fetish. It is somewhat strange that in the table of compositions of fuels on page 131 Welsh and Russian anthracites are mentioned, but no allusion is made to Pennsylvania anthracite which is the most important of all the deposits of that type of fuel at present known.

HENRY LEFFMANN.

BATS, MOSQUITOES AND DOLLARS. By Dr. Charles A. R. Campbell, with an introduction by Ernest Thompson Seton. xi–262 pages, numerous plates, small 8vo. Boston, The Stratford Company, 1925. Price, $3.

Many persons after reading this book will be inclined to ask whether the author is a prophet calling to us out of the wilderness or only a fakir. It would be unfair to Doctor Campbell to class him under the latter title, for his work is scientific and entirely sincere. He makes four postulates, of which the first will be accepted by all informed persons without question, namely, that the malaria mosquito is one of the worst enemies of mankind. Most mosquitoes are serious pests. Fortunately many species do not convey specific infections,

but the few that do this are surely deserving of efforts at extermination, relentlessly and energetically.

The main thesis of the book is that the common bat is the most efficient enemy of all mosquitoes (and presumably of other night-flying insects, but many of these are not so important) and, therefore, efforts should be made to protect bats and to encourage breeding. In addition, attention is called to the value of bat-manure as a fertilizer, but that is a secondary matter, in spite of the fact that Doctor Campbell makes it the basis of the somewhat startling termination of the title. Many investigations and experiments have been made by him, and there seems to be no doubt that infestation by mosquitoes can be eliminated by constructing bat-roosts, in such locations as will afford the animals opportunity to visit the infected areas. Incidentally, but very naturally, Doctor Campbell speaks of the use of the figure of the bat as an emblem of evil. Its peculiar and certainly not pleasing form and its night-flying habits have led to this attitude. "Angels and ministers of grace" are represented with the wings of birds, but it will be hopeless to expect any artist to represent such by the picture of a bat. It must not be forgotten that while the common bat of our northern climate is by reason of its size no formidable animal, much larger bats, probably blood-suckers, are common in warmer regions. Bats are generally feared, especially by women, who have a superstition that the animal has a tendency to entangle itself in the hair. There is apparently no instance of this known, but superstitions require no proof. They flourish in spite of reason and experience. In discussing the widespread influence of malaria, Doctor Campbell quotes the suggestion of Ramsey (the theologian, not the physicist) that Paul's "thorn in the flesh" was chronic malaria. It is a free-for-all guess. A prominent Philadelphia physician thought the trouble was chronic appendicitis. The description of Paul's conversion suggests epilepsy. The book is liberally illustrated by excellent plates, and the methods of protection of the bat and encouragement of its breeding are set forth clearly and explicitly. The other methods that have been used for mosquito extermination are considered. Doctor Campbell recognizes the value of the use of oil, but points out the trouble and expense that it involves. It seems likely that the plan advocated in the book will receive comparatively little acceptance at present, for it is radical, and a very large number of persons will object to encouraging the multiplication of bats, and will prefer to suffer the evils of mosquitoes. Such a position is unreasonable, but, as remarked above, popular fears, prejudices and superstitions are well-nigh ineradicable. The book ought to have a wide circulation, for outside of the validity of the author's suggestions, there is a large amount of valuable information concerning bats, their habits, struggles and life history. Incidentally we are told that they do not like "jazz." In this they show good taste.

The work is not limited to the mosquito question; the later chapters are devoted to the part played by the bedbug in disseminating small-pox. Doctor Campbell is happily not an anti-vaccinationist, but he is opposed to the theory that small-pox is disseminated through the air or by clothing or other objects that have come in contact with infected persons. He argues that the bedbug alone is responsible. This seems to be an extreme view, but there is no reason to doubt that a vicious blood-sucking insect, such as *Cimex lectularius*, can con-

vey many diseases. Other insects of the same habits must not be overlooked. The several species of lice, acari and fleas are important disseminators of disease. It was stated authoritatively that the epidemic of typhus fever in Serbia, a few years ago, was almost exclusively propagated by lice. The reviewer commends this book to the careful attention of sanitarians.

HENRY LEFFMANN.

THE FUNDAMENTALS OF PHYSICAL CHEMISTRY, for students of chemistry and related sciences. By Arnold Eucken, Technical High School, Breslau, Germany. Translated by Eric R. Jette and Dr. Victor K. LaMer, of Columbia University. xxiii–699 pages, illustrated, small 8vo. New York, McGraw-Hill Book Company, 1925. Price, $5.50.

A review of this book is no easy task. It may be safely assumed that from the pen of Professor Eucken can come only acceptable statements as to the present position of physical chemistry and that the translation of such statements into English will be carefully and correctly carried out by Messrs. Jette and LaMer. These features are, however, not the whole problem. Works on physical chemistry, originally written in English by authorities in the field, are now abundant and some good reason should be adduced for adding a translation, liable to be a doubtful medium of instruction. The translators are aware of this view, expressing in their preface the liability of critics to ask why the work has been done. They say that the essential characteristic of this book is the consideration of kinetics, thermodynamics and the quantum theory on the basis of experimental data as of equal importance in the development of the subject. The title of the German original is "Grundriss der physikalischen Chemie." A minor criticism might be made that "fundamentals" is not exactly the proper rendering, but the work can hardly be regarded as a mere "outline." It plunges at once *in medias res*, beginning with a chapter on "mathematico-physical" concepts. The author, indeed, in his preface to the second edition, frankly states his preference for a dominant mathematical treatment of the subject. Though bearing in its subtitle the claim as a student's manual, it must be said that the great majority of students would find it rather discouraging in comparison with some of the books now available, in which the experimental data are much more prominent, and the mathematical demonstrations incidental. This opinion is not to be construed as disparaging the work. The question is one of method in teaching and any method will be largely influenced by the material with which the teacher has to deal. German students probably in large part enter upon scientific courses with fairly thorough preparation in algebra and calculus, but much less proficiency is seen in the generality of students who enter on scientific courses in the United States.

After all, it is a serious question how far chemistry or its cognate subjects should be taught apart from the practical side. There seems to be a tendency (somewhat exemplified in this book) to promote a purely philosophic view of the science. It would be equally unwise to drop theoretic treatment. As was said years ago by a German chemist, it was the equal cultivation of theory and practice that enabled the German coal-tar industry to conquer the world, a conquest that was soon painfully appreciated when the World War broke out. So it would seem that manuals on physical chemistry should be pretty well filled

with illustrative experiments, associated with such mathematical explanations as are necessary to the preservation of the proper relation between theory and fact. Undoubtedly, teachers and research workers will differ much on what this proportion should be. Professor Eucken, as already noted, leans to a very large allowance of the mathematical phase. He alludes to the difference of opinion on this question that has been manifested by some of his colleagues, but he asserts that when "it is desired to make accessible to students the resources of physical chemistry, which are so useful for practical purposes, the task is made more difficult instead of easier for both teacher and pupil in the attempts to avoid mathematics." The validity of this assertion will depend much upon the teacher's ability to make clear the mathematical demonstration (not by any means a universal ability) and the pupil's preliminary training in mathematical abstraction.

The translators have done their work well. They understand the subject and have had the direct assistance of the author. The text is in excellent English. They have added a very large amount of additional statements. There is, however, no little conflict between the statements in the original work and these annotations, so that one gets the impression that some of the "fundamentals" of physical chemistry are in a rather "colloidal" form. The book is well worth attention by the advanced worker in physical chemistry.

HENRY LEFFMANN.

THE CHEMICAL ACTION OF ULTRAVIOLET RAYS. By Carleton Ellis and Alfred A. Wells, assisted by Norris Boehmer. 362 pages, 85 illustrations, 8vo. New York, The Chemical Catalog Company, 1925. Price, $5.

This is one of the most interesting books that the reviewer has seen for some time. Dealing as it does with a peculiar and comparatively unfamiliar field of physico-chemical phenomena, it presents a vast mass of facts in a clear and comprehensive form. The Newtonian discovery gave information as to the composite nature of white light and enabled scientists to explain the color phenomena in nature. Some of the Greek philosophers supposed that the rainbow was due to the effects of rain-drops on light, but in the absence of any knowledge of the composite nature of light they could give no explanation. It is curious to note the contrast between the Jewish and the Greek minds in relation to the rainbow. Early Jewish literature presents no speculation as to the nature of the bow; the word by which it is mentioned in the book of Genesis means "the weapon," and the significance of the appearance is that the deity has hung up the weapon as an earnest of the cessation of warfare against man. The Greek did not speculate on the teleologic significance of the appearance, but tried to discover its nature.

Newton was unaware of the existence of invisible rays, and since he used a rather large opening for the admission of the beam of light to the prism, he also overlooked the numerous dark lines, which cross the solar spectrum and indicate missing waves, so that solar light is really imperfect. In the early part of the nineteenth century, it was discovered beyond the violet and below the red are a long series of waves invisible to human eyes, but having noticeable chemical and physical effects. The work in hand is limited to the study of the chemical action of the rays beyond the violet. It seems that the infra-red field

is generally neglected, possibly because the phenomena to which the rays give rise are not so spectacular or so widely applicable to practical uses.

To any one who wishes to learn the information now at hand concerning the nature and effects of the ultra-violet rays, this book can be cordially recommended. A chapter on sources gives many ways in which the rays may be obtained. The important and striking uses, including therapeutics and hygiene, to which the rays have been put, have stimulated inventive talent and several excellent generators are now on the market. They are unfortunately still quite expensive. The most familiar depend on electrical discharges for action. High tension sparks from iron poles and discharges in mercury vapor are two of the most familiar. Inasmuch as the rays are almost all stopped by even very thin layers of ordinary glass, quartz tubes must be used and this adds considerable expense. Uviol glass which passes a considerable proportion of the rays is noted, but the glass lately introduced by the Corning works, which is dark purplish and cuts out almost all visible rays, thus transmitting a nearly pure beam of ultra-violet, seems to be not mentioned. The improvements in the manufacture of quartz glass are progressing encouragingly, and it is to be hoped that before long a complete quartz train for camera and photomicrographic work will be available at reasonable cost.

There is a certain amount of ultra-violet light of comparatively long waves emitted from the sky. It is stated in this book that the short waves emitted by the sun are mostly scattered and absorbed before reaching the earth, but the clear blue sky is generally a sufficiently abundant source of the rays as to require the use of yellow screens in taking pictures on Lumière plates. Wood showed that many leaves reflect large amounts of infra-red, and later observers have discovered that many flowers reflect notable amounts of ultra-violet that are present in ordinary outdoor light. There is possibly such a difference between these two reflections that the contrast between the flower and leaf is intensified to any insect which is capable of "seeing" the two classes.

This book deserves wide circulation. It is to be hoped that some one will take up the burden of writing a similar work on infra-red, thus bringing before the general scientific world the story of the "invisible lights."

HENRY LEFFMANN.

A SURVEY OF PHYSICS, a collection of lectures and essays. By Max Planck, translated by R. Jones and D. H. Williams. viii–184 pages, 7½ × 5 inches, cloth. New York, E. P. Dutton and Company, no date. Price, $2.40.

There is a small body of celebrated investigators who are at the forefront in the development of modern physical theories. Professor Planck is an outstanding figure among them. His studies in radiation and evolution of the quantum theory are now widely adopted as a satisfactory working hypothesis for certain phenomena for which the laws of Newton and those of Maxwell have proved inadequate. Both on account of a commanding position in a new path in physical science and an otherwise established leadership in the fields of mathematics and physics, his "survey" will meet with a warm welcome by all who are interested in the fundamentals of the subject.

The title plainly states the purport of the work. The volume embraces the collection in chronological order of eight lectures given on several occasions

during recent years, namely, The Unity of the Physical Universe, The Place of Modern Physics in the Mechanical View of Nature, New Paths of Physical Knowledge, Dynamical Laws and Statistical Laws, The Principle of Least Action, The Relation between Physical Theories, The Nature of Light, The Origin and Development of the Quantum Theory.

The present state of physical science is set forth with great clearness and leaves little doubt in the mind of the reader that the probability of bringing the phenomena of physics under one fundamental law is still remote and statistical laws at present dominate. The author frankly states that while "physics in its development has really advanced and we have learned to know Nature appreciably better each decade, . . . the latest and most mature results of research must be embraced in rigid and convenient definitions and by a suitable classification of subjects." "No physical theorem is at present beyond doubt, all and every physical truth is considered disputable. It often seems almost as if theoretical physics is about to be plunged again into chaos."

The translation is exceptionally well done, the phrasing shows no trace of its original German. LUCIEN E. PICOLET.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS. Report No. 215, Air Forces, Moments, and Damping of Model of Fleet Airship, *Shenandoah*. By A. F. Zahm, R. H. Smith, and F. A. Louden. 32 pages, illustrations, quarto. Washington, Government Printing Office, 1925. Price, fifteen cents.

To furnish data for the design of the fleet airship, *Shenandoah*, a model was made and tested in the 8 by 8 foot wind-tunnel for wind forces, moments, and damping, under conditions described in this report. The results are given for air of standard density, $\rho = .00237$ slug per cubic foot without L/v correction, and with but a brief discussion of the aerodynamic design features of the airship. This account is a slightly revised form of Report No. 195, prepared for the Bureau of Aeronautics, July 22, 1922, and by it submitted for publication to the National Advisory Committee for Aeronautics.

THE KINETIC THEORY OF GASES. By Eugene Bloch, Professor at the Lycee St. Louis, translated by P. A. Smith, B.A. (Oxon). xiv–178 pages, $7\frac{1}{2} \times 5$ inches, cloth. New York, E. P. Dutton and Company, no date. Price, $3.

The kinetic theory of gases is inseparably connected with the molecular theory of matter and from the laws of Dalton, Avogadro, and Gay-Lussac the theory has been established by the classic analysis of Maxwell and Clausius on the firm basis upon which it rests to-day. Since the essential feature of this theory is that heat is the kinetic energy of the molecules plus the potential energy of intermolecular attractions, the mathematical calculations to which it leads bear upon these motions to which, in the aggregate, observable properties are attributable. From a large number of such observed properties by the aid of equations of probability the theory is developed.

Without entering into the detail of calculations the deductions are presented in logical order in mathematical language. Among the extensive collection of researches of celebrated investigators may be mentioned: Laws of Perfect

Gases, Van der Waal's Equation of State, Mean Free Path, Equipartition of Energy, Thermodynamics and Statistical Mechanics, Entropy and Probability, Specific Heats of Solids, Theory of Quanta.

The publishers are rendering American students of science a real service in issuing a series of excellent translations of important foreign works on physical science. There are many such works still unavailable to American readers and the field is fertile and intellectually highly profitable. It is to be hoped that the publishers will find it expedient to make the list a long one.

LUCIEN E. PICOLET.

INTRODUCTION TO QUALITATIVE ORGANIC ANALYSIS. By Dr. Hermann Staudinger, Halle a.S. Translated by Walter T. K. Braunholtz, Ph.D. xvi–112 pages, small 8vo. New York, D. Van Nostrand Company, 1925. Price, $2.50.

This is not an elementary treatise, at least not of the usual form. It deals rather with classes of compounds than with individual ones. Its plan is to enable the student to detect organic compounds as they may occur in mixtures and identify them without elementary analyses. The field is a large one and one of the most difficult tasks is to determine what shall not be included. The author, in his preface, states that most books introductory to organic chemistry from the practical side lay stress on synthetic work. This is, indeed, the most attractive to students, and considering what organic synthesis has accomplished and how necessary to the welfare of Britain and America is the cultivation of this phase, most teachers will be inclined to give preference to that type of book. The work is of German origin, and Germany is the country in which the activity in all departments of organic chemistry has been most successful and extensive. It is natural, therefore, that teachers should wish to find new lines of effort.

So far as the general character of the work is concerned, there is no need for criticism. The author's qualifications are unquestioned, and the translator has done his task well. The book has no index, which is a serious defect. There is, it is true, an elaborate table of contents, but a comprehensive index would have been of much aid. The table of contents could then have been much abbreviated.

HENRY LEFFMANN.

TABLES AND GRAPHS FOR FACILITATING THE COMPUTATION OF SPECTRAL-ENERGY DISTRIBUTION BY PLANCK'S FORMULA. By M. Katherine Frehafer, Associate Physicist, and Chester L. Snow, Draftsman, Bureau of Standards, assisted by Harry J. Keegan. Miscellaneous publication of the Bureau of Standards, No. 56, for sale by the Superintendent of Documents, Government Printing Office, Washington, D. C. Price, thirty-five cents.

The spectral-energy distribution of an illuminant is readily determined by matching it in color with a so-called "black body." The temperature to which this "black body" is brought in order to produce the color match is readily determined with suitable laboratory equipment and the energy distribution in the visible spectrum is readily computed from the Planck formula which expresses energy in terms of wave-length and absolute temperature. The tables and

graphs have been devised to save the considerable time and labor entailed in computations of this exponential formula.

The series consists of seven sheets 24 in. × 19 in. as follows: Sheet I, Text; Sheets II, III, IV, Graphs 1000 to 5000 K; Sheet V, 5000 to 8000 K; Sheet VI, 8000 to 24,000 K; Sheet VII, Tables. The text contains an explanation of the formulas employed and detailed directions for the use of the tables computed from them. The construction of the graphs, the scale adopted and their limits of error are also explained. The mechanical execution of these charts is of the superior quality characteristic of the Bureau of Standards and particularly the draftsmanship which is of a high order of excellence.

LUCIEN E. PICOLET.

PUBLICATIONS RECEIVED.

The Elements of Chemistry, by William Foster, A.M., Ph.D. 576 pages, illustrations, 8vo. New York, D. Van Nostrand Company, 1925. Price, $2.

Bats, Mosquitoes, and Dollars, by Dr. Charles A. R. Campbell. 262 pages, illustrations, plates, 8vo. Boston, The Stratford Company, 1925. Price, $3.

The Chemical Action of Ultraviolet Rays, by Carleton Ellis and Alfred A. Wells, assisted by Norris Boehmer. 362 pages, illustrations, 8vo. New York, The Chemical Catalog Company, 1925. Price, $5.

Fundamentals of Physical Chemistry for Students of Chemistry and Related Sciences, by Dr. Arnold Eucken, translated and adapted from the second German edition by Eric R. Jette, Ph.D., and Victor K. LaMer, Ph.D. 699 pages, illustrations, 12mo. New York, McGraw-Hill Book Company, 1925. Price, $5.50.

Ostwald-Luther Hand- und Hilfsbuch zur Ausführung physikochemischer Messungen unter Mitwirkung, von W. Bothe, W. Gerlach, R. Gross, H. v. Halban, R. Luther, F. Paneth, F. Weigert, herausgegeben von C. Drucker. Edition 4. 814 pages, illustrations, 8vo. Leipzig, Akademische Verlagsgesellschaft m. b. H., 1925. Price, $8.35.

Aprovechamiento de las Energías Naturales. Estudio descriptivo y razonado de los modernos métodos que la técnica y la economía aconsejan para el aprovechamiento más perfecto posible de las fuentes naturales de energía, por Juan Gelpi Blanco. 283 pages, illustrations, 8vo. Barcelona, Libreria Subirana, 1924. Price, 24 pesetas.

Proceedings of the First International Congress for Applied Mechanics, Delft, 1924, edited by Prof. Ir. C. B. Biezeno and Prof. Dr. J. M. Burgers. 460 pages, illustrations, plates, quarto. Delft, J. Woltman, Jr., 1925.

XIXe Salon International de Photographie, Ce Salon organisé par la Societé Française de Photographie et le Photo Club de Paris, s'est tenu en l'Hôtel de la Société Française de Photographie, 51 rue de Clichy, Paris, du 3 au 14 Octobre, 1924. 48 photogravures, quarto. Paris, Society, 1924.

Étude sur l'Écoulement des Fluides en Général, par Mm. A. Lebrasseur et F. d'Espine, avec une note au sujet de la loi d'Ombeck par Mm. Lebrasseur et Henzi et deux notes de M. Lebrasseur sur les incidents de route et les pertes de charges dans les faisceaux tubulaires. Extrait des numeros de Chaleur et Industrie. 73 pages, illustrations, plates, quarto. Paris, 1925. Price, 14 francs.

Manual of Organon, the philosophy and science of organing the art of management, by Charles W. Gremple, B.Sc., M.E. 32 pages, 12mo. New York, Organon Lyceum, 1925. Price, $2.

Technical Books of 1924. A selection compiled by Donald Hendry, head of the applied science reference department, Pratt Institute Free Library. 27 pages, 12mo. New York, Pratt Institute, 1925.

National Advisory Committee for Aeronautics: Technical Notes, No. 226, Wind-tunnel Tests of Fuselages and Wind-shields, by Edward P. Warner. 6 pages, diagrams, quarto. Technical Notes, No. 227, Determination and Classification of the Aerodynamic Properties of Wing Sections, by Max M. Munk. 22 pages, quarto. Washington, Committee, 1925.

United States Geological Survey: Special Publication No. 117, The Earth's Magnetism, by Daniel L. Hazard. 52 pages, illustrations, plates, 8vo. Washington, Government Printing Office, 1925. Price, fifteen cents.

United States Department of Commerce: Bureau of Mines Technical Paper No. 373, The Pyrotannic Acid Method for the Quantitative Determination of Carbon Monoxide in Blood and in Air, by R. R. Sayers and W. P. Yant. 18 pages, plates, 8vo. Washington, Government Printing Office, 1925. Price, five cents.

United States Department of Commerce: Technologic Paper of the Bureau of Standards No. 290, Relation between Heating Value of Gas and Its Usefulness to the Consumer, by E. R. Weaver. 116 pages, diagrams, quarto. No. 295, Initial Temperature and Mass Effects in Quenching, by H. J. French and O. Z. Klopsch. 28 pages, diagrams, quarto. Scientific Papers, No. 509, Alternating Current Distribution in Cylindrical Conductors, by Chester Snow. 57 pages, quarto. No. 510, Effect of Wear on the Magnetic Properties and Tensile Strength of Steel Wire, by R. L. Sanford and Others. 4 pages, illustrations, plates, quarto. Washington, Bureau of Standards, 1925.

Petroleum Refining. A brief description of organized and experienced engineering and construction service, as applied to the petroleum industry. 25 pages. illustrations, quarto. The J. G. White Engineering Corporation, New York.

Photo-resistance Effect for Metals at Low Temperatures. R. S. BARTLETT. (*Phys. Rev.*, Aug., 1925.)—Waterman has developed a theory of the resistance of metals according to which at very low temperatures their resistances should change under illumination. Thin films of gold, silver, platinum, palladium, bismuth, tellurium and copper were prepared by the method of cathode sputtering. In the case of copper, hydrogen was used in the discharge tube to avoid oxidation. All the films were aged before their resistance under light was measured. Light from a quartz-mercury vapor lamp fell upon the films. The resistance was measured by a potentiometer method by which a change of one part in three million could be detected. Care was taken to obviate any change of resistance due to a rise of temperature caused by the incident radiation.

Films of gold and silver showed no change of resistance due to

light. Platinum displayed a mere trace of change. "Copper showed a very definite and instantaneous decrease in resistance under the influence of light for the thinnest films used, 1.5×10^{-6} cm. in thickness. The decrease was found to be between 1.1 and 1.6 parts in 10^6 for different films. Palladium and bismuth showed much larger changes, amounting to 14 and 16 parts in 10^6, respectively. Tellurium films, sensitive at room temperature, showed a considerable increase in sensitivity at liquid air temperature, in agreement with the prediction of theory. These results conform rather well with predictions. The evidence appears, then, to support Waterman's prediction of a light sensitivity for metals at low temperatures." The behavior of bismuth was erratic.

The lowest temperature was that of liquid air. The author holds that the size of the changes in resistance could be augmented by the use of lower temperatures, thinner films and shorter wave-lengths.

G. F. S.

Mechanical Pressure Tester for Determining Maturity of Fruit. (U. S. Department of Agriculture, Press Service.)—The maturity of fruit at picking time largely determines its dessert quality as well as its storage or transportation possibilities. Allowing the fruit to become too mature on the tree results in a product which will not carry to the market, whereas picking the fruit in a too mature condition results in an almost inedible product.

Pressure on the fruit with the thumb to determine the maturity has probably been practised as long as fruits have been eaten by man, but this method is too indefinite for modern demands of fruit dealers and handlers. This need for definite picking standards for certain fruits which will allow them to be left on the tree as long as possible and at the same time assure their carrying through to market in satisfactory condition, has led the Department of Agriculture to devise an improved type of mechanical pressure tester, a description of which is given in Department Circular 350, just issued by the department.

The pressure tester is somewhat on the order of an automobile tire pressure gauge, is of convenient portable size, and easy to use. A protruding plunger of the tester, placed against the flesh of an apple or other fruit, penetrates the flesh, recording the maximum pressure required to penetrate the fruit to a given distance, thus giving an indication of its maturity. By means of the tester and storage experiments, definite picking standards are being worked out for the various fruits. The rate of softening of fruit in storage and the firmness of fruit in different stages of storage maturity are also being studied.

When these schedules have been completed, the fruit grower will know at what pressure he should pick his fruit to enable it to stand the storage and transportation conditions expected of it. The fruit dealer will be able to test his fruit in storage and determine the remaining length of time he can expect it to keep in firm condition.

CURRENT TOPICS.

The New Ideas in Meteorology. G. C. SIMPSON. (*Nature*, Sept. 5, 1925.)—In this, the presidential address before Section A at the recent meeting of the British Association, attention is directed to the great advances made in the physical sciences during the first quarter of the present century and then this comment is added: "Meteorology, though a child of applied mathematics and physics, has scarcely been touched by the epoch-making discoveries in the house of its parents. The quantum has found no place in our theories of the mechanism of the atmosphere; a knowledge of the structure of the atom has not helped us to understand the physics of the air as we deal with it in meteorology; the relationship between mass and charge, the invariability of the velocity of light, four-dimensional space and all the other new conceptions which have been responsible for the advance of physics, have been of no help to meteorologists in their especial branch of science." In spite of this fact and in spite of the additional disadvantage that physicists have been so engrossed in their own fields of research as to disregard meteorology, "there has been in the last twenty-five years as fundamental a revolution in our ideas of the atmosphere as has taken place in our ideas of electricity and matter."

Teisserenc de Bort in 1898 began to send up his *ballons-sondes* carrying recording instruments into unexplored regions of the atmosphere. His success and that of those who followed the path he had blazed led to the formation of an International Commission to guide investigation of the upper atmosphere. Expeditions have studied this region over equatorial Africa, over the poles and over the oceans lying between. From the results of this intensive study the following has been learned: "The atmosphere, which itself is an extremely thin layer of air, is composed of two shells surrounding the earth. In the lower of these shells, called the troposphere, the temperature decreases as one rises in the atmosphere, and the air is warmer over the equator than over the poles at corresponding heights. In the upper shell, called the stratosphere, the temperature conditions are entirely different. There is little or no change in temperature with height, and the horizontal change of temperature is reversed, the temperature at the same height in the stratosphere decreasing as one passes from the poles to the equator. At the earth's surface the mean annual temperature near the equator is 27° C., and at the poles −23° C., *i.e.*, the equator is 50° C. warmer than the poles. At twenty kilometres above the surface the temperature over the equator is −80° C., and over the poles −30° C.; that is, the temperature difference between the equator and the poles is the

same in amount at the surface and at a height of twenty kilometres, but in the former case it is the equator which is warmer, while in the latter it is the polar regions—a truly remarkable reversal." Moreover the surface of demarcation between the two layers, called the "tropopause" by Sir Napier Shaw, is very distinct, somewhat like that below a layer of oil floating on water. This tropopause is higher above the earth at the equator than at the poles, the troposphere being thicker at the equator than in higher latitudes. "At the equator it is nearly twenty kilometres thick, while at the poles it thins down to a layer less than six kilometres thick in the summer and less than four in the winter."

With the omission of the two kilometres of air nearest to the earth, where the character of the surface, the time of day and the season of the year make a difference, the "lapse rate," the rate of decrease of temperature with height, is the same all over the earth. It amounts to 5.6° C. for each kilometre of ascent for the air lying from two to four kilometres above the earth. The rate increases with elevation and becomes as much as 7.1° C. toward the highest parts of the troposphere.

It is likely that most people have thought ascending and descending currents of air to be normal occurrences. The author, however, maintains that they are rare, being due to exceptional combinations of conditions. Certain others of our cherished beliefs must likewise go by the board. "Observations have shown that there is no direct flow of air from the equator to the poles in the upper atmosphere, and measurements of temperature in cyclones and anticyclones have shown that the former are not warm and the latter are not cold." It seems, furthermore, that we must accustom ourselves to the existence in the atmosphere of surfaces of discontinuity such as that separating the troposphere from the stratosphere. In addition to this V. Bjerkenes holds that there are two other great, permanent surfaces of discontinuity. One of these lies between the trade winds and the anti-trade winds moving above them. The other separates the cooled air drifting toward the west away from the polar regions on one side of the boundary from the air in middle latitude moving from west to east on the other side. Besides these permanent surfaces there are temporary ones that form and disappear. "The forecaster searches his chart for indications of the surfaces of discontinuity, and examines the characteristics of the air masses to see whether they are of polar or equatorial origin. In this way he is able to determine the structure of the cyclone and whether it is developing or dying. Having determined where the surfaces of discontinuity are situated, he is able to say when rain may be expected, and he knows what weather changes will accompany the passage of each surface of discontinuity as it moves over the surface of the land. . . . Unfortunately the processes which take place in the atmosphere are extremely complicated, and perfect forecasts are still far from being attained."

G. F. S.

The Connection between the Attached Gas and Changes in the Resistance of an Iron and of a Platinum Wire in a High Vacuum. H. KLEINE. (*Zeit. f. Physik,* **33**, Aug. 8, 1925.)—While few regions of physics have been so thoroughly explored as that of the electrical resistance of metallic conductors, it is nevertheless true that little work has been done on the connection between such resistance and the amount of gas attached to the metal. In the older literature on the subject palladium seems alone to have been investigated, having attracted attention by its unusual appetite for hydrogen. Even after the possibility of attaining and keeping high vacua had come, we find Meyer in 1911 in conflict with Suhrman, 1923, the former maintaining that the removal of gas causes a decrease of resistance, while the latter obtained an increase.

In studying this problem it must not be forgotten that gas is attached not only to the wire under investigation, but to the walls of the apparatus as well. To drive the gases away from the glass parts not immediately surrounding the wire, the method of bombardment by ions was applied by the use of the discharge without electrodes. The tube containing the wire was, however, heated by an electric furnace that brought the temperature up to 450° C. During the release of the gases from the glass it is not impossible for the wire to appropriate to itself an additional quantity from their mere proximity. It should also be remembered that part of the wire's load of gas is adsorbed upon its surface, while the remainder is absorbed and is more intimately connected with its molecular structure.

With an iron wire 16 cm. long and .24 mm. in diameter removal of air from the tube surrounding the wire was followed by a decrease in the resistance. Upon the readmittance of air the resistance at once grew larger. A footnote mentions that light produced a slight increase of resistance, provided that the pressure of the air in the tube was below 8 mm. of mercury.

A more extensive series of experiments was undertaken with a platinum wire 17 cm. long and .2 mm. in diameter. With this metal any change of resistance that seemed to be associated with change of air pressure was too small to throw any light on the connection between attached gas and resistance. A more frontal attack was made by driving the gas out of the wire by heating. Before this was done the glass parts were freed from gas until the pump maintained a vacuum of from 10^{-7} to 10^{-9} mm. Thus any changes following upon the heating of the wire must be ascribed to effects on or within the wire itself to the exclusion of the rest of the apparatus. The course of the experiments was as follows: By a contact device operated by a clock the current was sent for a few seconds through the wire and then was interrupted. This succession of heatings was continued until the current had flowed in the wire for forty-five minutes or longer. At some time during this interval the air pressure was measured. This, of course, proved to be much higher than before the gas began to escape from the wire. In a series of heatings amounting in

all to about twenty hours, it fell from $.65 \times 10^{-5}$ to $.28 \times 10^{-5}$ mm. of mercury, though at the beginning after one and one-half hours it rose to more than twice its earliest value. After the current had heated the wire for forty-five minutes or more, it was stopped and the wire cooled to room temperature. The resistance was then measured, and the heating of the wire was resumed. This series of operations was continued until after twenty hours of heating the resistance of the wire had attained a fairly unchanging value. Before the wire was heated its resistance was .9131 ohm and at the end of 1240 minutes of heating it had diminished about 12 per cent. to .8065 ohm. The total quantity of gas driven off would occupy at normal pressure twenty times the volume of the wire from which it escaped. The temperature coefficient of the resistance of the platinum wire was measured for the temperature interval from 16° to 45° C., but no conclusion could be drawn as to a possible effect of the attached gas upon the value of the coefficient.

Again with platinum as with iron, light increased the resistance, and no more intense light than that of an incandescent lamp used to heat the oil bath was needed. After from five to eight minutes the additional resistance due to its light attained a constant value independent of the prevailing temperature and of the pressure.

G. F. S.

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